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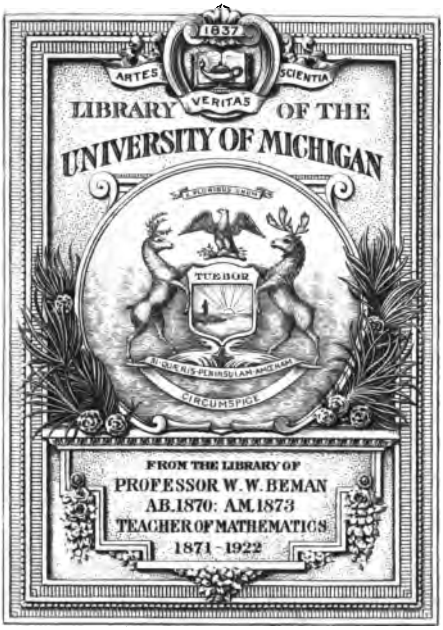

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THE
ELEMENTS
OF
TRIGONOMETRY.

CONTAINING

The Properties, Relations, and Calculations of
Sines, Tangents, Secants, &c.

The Doctrine of the Sphere, and the
Principles of Plain and Spherical
Trigonometry.

All plainly and clearly demonſtrated.

The Third Edition, with Additions and Corrections.

Quicquid in Aſtronomicis, Geographicis, vel Nauticis efficiendum,
Trigonometriæ attribuendum eſt.

LONDON:
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W. W. Beman
gt.
6-13-1923

THE

PREFACE.

THE uſe of Trigonometry *is ſo great in all the parts of the mathematics, that he muſt have made a very little progreſs therein, who is not ſenſible of it. Its help is called in upon every occaſion, and its great ſervice is clearly apparent in calculations of all ſorts, both upon the earth, upon the ſeas, and in the heavens. By this, the diſtances of objects upon the earth may be certainly known, if they can but be ſeen; though we cannot come near to meaſure them: likewiſe the geographical diſtances of places on the earth; and their ſeveral poſitions to one another. Navigation depends entirely upon it. Surveying and dialling owe their greateſt exactneſs to it. It is of ſingular ſervice in military affairs: and* Mars, *without this, might live peaceably at home. Upon the wings of Trigonometry (as* Plato *ſays) we mount up from the earth to the heavens, meaſure the diſtances of all the ſtars, and range them in their proper order. And without it* Urania*'s ſons may throw aſide their inſtruments, their books and tables; or rather, without this, they would never have had ſuch things; and conſequently mankind had remained utterly ignorant of this moſt beautiful ſyſtem of the world. In ſhort, the art of Trigonometry is of ſuch univerſal uſe and extent, that it would be an endleſs taſk to enumerate all the various purpoſes to which it is ſubſervient; and the moſt important branches of knowledge would be loſt and uſeleſs if we wanted it.*

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This makes me wonder that no body has given us an entire ſyſtem of this art; which ſhall contain all the principles, *as well as the rules of* practice. *Inſtead of that, moſt authors content themſelves with little more than explaining the common caſes of ſolving triangles, and fill up their books with heaps of examples, to ſhew the uſe and application thereof: whilſt they dip very little into the theory, and leave the principles thereof very much in the dark.*

In this treatiſe I have ventured to lay down the whole both theory *and* practice; *and to take in all things of any conſequence that any way belong to the ſubject. And here I account all theſe properties of triangles or other figures that have any relation to the meaſuring their angles by degrees, or by ſines, and tangents, &c. to belong to* Trigonometry, *as their proper ſubject; referring their other properties to* Geometry, *which have no ſuch regard to the meaſures of their angles.*

In purſuance of this deſign, I have in the firſt book laid down in a few propoſitions, the relations of ſines, coſines, &c. of arches; likewiſe of the double arches, and of the ſums and differences of arches. Among which there are a few propoſitions, ſo extenſive and general, as to comprehend a great number of particular propoſitions, evident only by inſpection.

Then I have given the method of calculating natural *and* artificial ſines, tangents, *&c. of any arch: and from thence thoſe of their* multiple arches. *And here I have been obliged to have recourſe to the method of* fluxions, *and the method of* increments; *there being no other poſſible way to effect it. Which if any of my readers happen not to underſtand, they may paſs by the inveſtigations,*

tions, and make equal uſe of the concluſions, nevertheleſs. But for further ſatisfaction, ſee my Method of Increments.

Then you have a few general propoſitions concerning angular ſections, inſcription *of* polygons, *and the* properties of chords *inſcribed in a circle.*

In the ſecond book you have plain Trigonometry, *with the ſolution of all the* caſes, *ſeveral ways. Right-angled triangles are reſolved arithmetically, logarithmically, and algebraically; and oblique ones logarithmically and algebraically. By the firſt method all the caſes of right-angled triangles are reſolved by common arithmetic, without any books or tables whatever. And this comes very near the truth, and is ſufficiently exact for common uſes: and is no further burthenſome to the memory, than getting by heart two or three proportions, with certain fixed numbers; and the* 47. I. Euclid. *And the ſame way may oblique triangles be reſolved, by letting fall a perpendicular.*

In the third book you have all that is material in the doctrine *of the* ſphere; *likewiſe the properties of* ſpherical triangles; *and the principles of* ſpherical Trigonometry; *in which are ſome compound or general propoſitions, including ſeveral particular ones. Among theſe there are ſome that I have taken the liberty to demonſtrate algebraically, to avoid a more prolix method of demonſtration. Then follows the ſolution of all the caſes of ſpherical triangles; in oblique ones both logarithmically and algebraically.*

In the firſt ſection of the firſt book, I have inſerted ſeveral ſcholia *here and there: the uſe of which to Algebraiſts, will be too evident to be further*

ther inſiſted on. And for the ſame reaſon it was, that I inſerted the ſolution of the caſes algebraically.

All theſe things being laid down here in this book, in the ſhorteſt manner poſſible; I think I may venture to entitle it The Elements of Trigonometry. I *won't ſay that I have quite exhauſted the ſubject, but I am of opinion that I have omitted little or nothing of any conſequence.*

I think it needleſs to ſhew the application of Trigonometry to any part of mathematics: that being very eaſy for the reader to do himſelf; eſpecially as all authors abound in theſe things.

If what I have done meet with general approbation, I have my end; if not, I can only ſay,

In magnis voluiſſe ſat eſt.

W. Emerſon.

P. S. *In the firſt edition of this book, having delivered the theory of Trigonometry, with all the practical rules and canons for calculation; I thought it ſufficient to anſwer all the purpoſes that could be deſired. And that nothing may be wanting even to the loweſt claſs of readers, I have given the ſolutions of all the caſes of triangles in numbers. I have alſo made many alterations and additions in ſeveral parts of the book, and have added ſeveral propoſitions in ſpherics, which I have put at the end of the laſt ſection; becauſe they could not well be demonſtrated, without the propoſitions in that ſection; all which now makes the book entirely complete.*

W. E.

THE CONTENTS.

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Characters uſed in this book.

+ Addition.
— Subtraction, of the following quantity.
× Multiplied by.
= Equal to.
⊥ Perpendicular to.
∥ Parallel to.
: : Proportion.
> Greater than.
< Leſſer than.
∠ Angle.
∽ The difference.
s. ſine ; coſ. coſine ; T. tangent ; cot. cotangent ; ſec. ſecant ; coſec. coſecant ; verſ. verſed ſine ; ſup. ſupplement ; ſeg. ſegments ; diff. difference.

THE
ELEMENTS
OF
TRIGONOMETRY.

DEFINITIONS.

1. AN *arch* of a circle is any part of its circumference, as AB. A quarter of the circumference is called a quadrant, as AD; and the half a ſemi-circle, as ADI. FIG. 1.

2. The *radius* is a line drawn from the center to the circumference, as CA.

3. The *complement* of an arch is what it wants of a quadrant; BD is the complement of AB.

4. The *ſupplement* is what it wants of a ſemicircle, as BI is the ſupplement of AB.

Cor. 1. *The difference of two arches is = difference of their ſupplements.*

Cor. 2. *The complement of half an arch is equal to half the ſupplement of the whole arch.* For let Q be a quadrant and A an arch; then $Q - \frac{1}{2}A$ is the complement of half the arch, and $2Q - A$ is the ſupplement of the whole, and $Q - \frac{1}{2}A$ is half the ſupplement of the whole.

Cor. 3. *The difference of an arch and its complement is equal to the complement of twice the arch,* $A \backsim Q - A$ is the difference of the arch and its complement, and $2A \backsim Q$ is the complement of twice the arch, but $A \backsim Q - A = 2A \backsim Q$.

FIG. 1. 5. The *cord* or *ſubtenſe* of an arch AB, is a right line AB drawn between the extreme points of the arch.

6. The *ſine* (or right ſine) of an arch AB, is a right line BF drawn from one end of the arch perpendicular to the diameter paſſing thro' the other end of it.

Cor. *The ſine of an arch is half the cord of the double arch.*

7. The *coſine* of an arch is the diſtance between ~~the center and the~~ ſine, as CF.

8. The *verſed ſine* of an arch AB, is the part of the diameter, between the beginning of the arch and the ſine, as AF.

Cor. The *verſed ſine* + *coſine* = *radius.*

9. The *coverſed ſine* is the line DE, between the end of the quadrant AD, and BE perpendicular to DC.

10. The *tangent* of an arch BA is the line AG drawn from one end A of the arch perpendicular to the diameter, till it cuts the line CBG drawn thro' the other end of the arch.

11. The *cotangent* of an arch AB is the line DH drawn perpendicular to DC at the end of the quadrant AD, till it meet the line CBH drawn thro' the top of the arch AB.

12. The *ſemitangent* of an arch AB, is the line KC intercepted between the center and the line IK paſſing thro' the end B of the arch.

Cor. *The ſemitangent of an arch is = tangent of half the arch.* For if AT = TB, then ∠ACT = ∠AIB. Whence CK = AR.

13. The *ſecant* of an arch AB is the line CG drawn from the center of the circle, through the top of the arch AB, till it meets the tangent AG.

14. The *coſecant* of an arch AB is the line CH, drawn from the center, through the end B of the arch, till it meets the cotangent DH in H.

Cor. 1. *Hence the coſine, cotangent, coſecant, coverſed-ſine of an arch, is equal to the ſine, tangent, ſecant, verſed ſine, of its complement reſpectively.* FIG. 1.

For the ſine EB of the arch DB (the complement of AB) is = CF, the coſine of AB. Alſo DH, the cotangent of AB, is the tangent of its complement DB. Alſo CH, the coſecant of AB = ſecant of its complement DB. And laſtly DE, the coverſed ſine of AB, is the verſed ſine of its complement DB.

Cor. 2. *A ſine, tangent and ſecant, coſine, cotangent, coſecant, and coverſed ſine, are common to two arches which are the ſupplements of each other.* For theſe lines belong as well to the arch IB, as to AB.

15. A *degree* is the 360th part of the circumference of every circle; and therefore a ſemicircle contains 180 degrees, and a quadrant 90°. Alſo the 60th part of a degree is called a minute, the 60th part of a minute a ſecond, &c.

16. An *angle* ACB is the inclination of two lines AC, CB; and the meaſure of that angle is the degrees in the arch AB.

Note, The middle letter ſtands at the angle.

SCHOLIUM.

Tho' the coſine, tangent, cotangent and ſecant are common to two arches which are the ſupplements to each other; yet in an arch greater than a quadrant, theſe lines are truly negative, and muſt be ſo eſteemed, becauſe they are drawn the contrary way to thoſe in an arch leſs than a quadrant. And therefore the ſine, coverſed ſine, and coſecant, are always affirmative in all arches leſs than a ſemicircle, but negative in greater arches. And in general, the ſines of all arches in the firſt and ſecond quadrants are affirmative, in the third and fourth negative. The coſines in the firſt and fourth quadrants are affirma-

tive,

FIG. 1. tive, in the ſecond and third negative. The tangents in the firſt and third quadrants are affirmative, in the ſecond and fourth negative. The cotangents in the firſt and third are affirmative, in the ſecond and fourth negative. The ſecants in the firſt and fourth are affirmative, in the ſecond and third negative. The coſecants in the firſt and ſecond are affirmative, in the third and fourth are negative. All verſed ſines are affirmative: moreover, ſines, tangents, &c, of negative arches are contrary in their ſigns to thoſe of affirmative arches.

BOOK I.

The properties of ſines, tangents, ſecants, &c, of arches; the calculation of them, both natural and artificial; and alſo thoſe of multiple arches; angular ſections, &c.

SECT. I.

The relation of the ſines, tangents and ſecants of arches.

PROP. I.

In any arch, theſe are reſpectively proportional.

FIG. I.

Radius : *Sine* : *Coſine* ::
Secant : *Tangent* : *Radius* ::
Coſecant : *Radius* : *Cotangent*.

THE meaning of the propoſition is, that any four quantities lying in form of a rectangle are proportional. As thus; *Rad*: *Sine* :: *Secant* : *Tangent*. Or *Rad* : *Coſine* :: *Secant* : *Radius*. Or *Tangent* : *Radius* :: *Radius* to *Cotangent*. Or *Cotangent* : *Coſine* :: *Coſecant* : *Radius*. And ſo of others. The ſame is to be underſtood of the following propoſitions. Likewiſe if you have ſeveral quantities placed thus, A : B : C :: D : E : F; the meaning is, that A : B :: D : E, or A : C :: D : F, or B : C :: E : F, &c; or laſtly, if A : B :: C : D :: E : F, &c; it ſignifies, that A : B :: C : D, or A : B :: E : F, or C : D :: E : F; and ſo of others.

For in the ſimilar triangles BFC, GAC, CDH; BC : BF : CF :: GC : GA : AC :: CH : CD : DH.

Q. E. D.

Cor. 1. *Radius ſquare* = *ſine ſquare* + *coſine ſquare* = *ſecant ſquare* — *tangent ſquare* = *coſecant*

FIG. 1. *ſquare — cotangent ſquare.* For the triangles BFC, GAC, CDH are right angled.

Cor. 2. *Radius ſquare = tangent × cotangent = coſine × ſecant = ſine × coſecant.*

Cor. 3. *In any arches, the ſine is as the coſine × tangent, or as tangent divided by ſecant, or as coſine divided by cotangent, or reciprocally as the coſecant.*

Cor. 4. *Tangent is as the ſine × ſecant, or as* $\frac{\textit{ſine}}{\textit{coſine}}$, *or as* $\frac{\textit{ſecant}}{\textit{coſecant}}$, *or reciprocally as the cotangent.*

Cor. 5. *Secant is as tangent × coſecant, or as* $\frac{\textit{tangent}}{\textit{ſine}}$, *or as* $\frac{\textit{coſecant}}{\textit{cotangent}}$, *or reciprocally as the coſine.*

Cor. 6. *Coſine is reciprocally as the ſecant, cotangent reciprocally as the tangent, coſecant reciprocally as the ſine.*

Cor. 7. *The ſum of the ſquares of the ſine and verſed ſine of an arch = ſquare of the cord of that arch = 4 times the ſquare of the ſine of half the arch.* For AB = 2AL, and the triangle ABF is right angled.

Cor. 8. *The cord of an arch is a mean proportional between the verſed ſine and diameter.* For AF, AB, AI are continually proportional.

SCHOLIUM.

Let A = any arch. t = tangent. σ = coſecant.
r = radius. τ = cotangent. v = verſed ſine.
s = ſine. $ſ$ = ſecant. V = verſ. ſ. ſuppl.
c = coſine.

Then

$$\text{I. } s=\sqrt{rr-cc}=\frac{ct}{r}=\frac{tr}{ſ}=\frac{tr}{\sqrt{rr+tt}}=\frac{rc}{\tau}=\frac{rr}{\sqrt{rr+\tau\tau}}$$
$$=\frac{rr}{\sigma}=\frac{ſc}{\sigma}=\frac{t\tau}{\sigma}=\frac{r}{ſ}\sqrt{ſſ-rr}=\sqrt{v\mathrm{V}}=\sqrt{2rv-vv}.$$

II.

FIG. 1.

II. $c = r - v = V - r = \sqrt{rr - ss} = \frac{rs}{t} = \frac{rr}{\sqrt{rr + tt}} = \frac{rr}{ſ} = \frac{t\tau}{ſ} = \frac{s\sigma}{ſ} = \frac{s\tau}{r} = \frac{r\tau}{\sigma} = \frac{r\tau}{\sqrt{rr + \tau\tau}} = \frac{r}{\sigma}\sqrt{\sigma^2 - rr}.$

III. $t = \frac{rs}{c} = \frac{rs}{\sqrt{rr - ss}} = \frac{r}{c}\sqrt{rr - cc} = \frac{rr}{\tau} = \frac{rr}{\sqrt{\sigma^2 - r^2}} = \frac{cſ}{\tau} = \sqrt{ſ^2 - rr} = \frac{s\sigma}{\tau} = \frac{r\sqrt{2rv - vv}}{r - v} = r\frac{\sqrt{2rV - V^2}}{V - r}.$

IV. $\tau = \sqrt{\sigma\sigma - rr} = \frac{rr}{t} = \frac{s\sigma}{t} = \frac{cſ}{t} = \frac{rc}{s} = \frac{r}{s}\sqrt{rr - ss} = \frac{rc}{\sqrt{rr - cc}} = \frac{rr}{\sqrt{ſſ - rr}} = \frac{r - v}{\sqrt{2rv - vv}}r = \frac{V - r}{\sqrt{2rV - V^2}}r.$

V. $ſ = \sqrt{rr + tt} = \frac{rr}{c} = \frac{rt}{s} = \frac{r^3}{s\tau} = \frac{r\sigma}{\tau} = \frac{t\tau}{c} = \frac{s\sigma}{c} = \frac{rr}{\sqrt{rr - ss}} = \frac{r}{t}\sqrt{rr + \tau\tau} = \frac{r\sigma}{\sqrt{\sigma\sigma - rr}} = \frac{rr}{r - v} = \frac{rr}{V - r}.$

VI. $\sigma = \sqrt{rr + \tau\tau} = \frac{rr}{s} = \frac{rſ}{t} = \frac{r\tau}{c} = \frac{r^3}{ct} = \frac{t\tau}{s} = \frac{cſ}{s} = \frac{rr}{\sqrt{rr - cc}} = \frac{r}{t}\sqrt{rr + tt} = \frac{rſ}{\sqrt{ſſ - rr}} = \frac{rr}{\sqrt{2rv - vv}}.$

VII. $v = 2r - V = r - c = r - \sqrt{rr - ss} = r - \frac{rr}{\sqrt{rr + tt}} = r - \frac{r\tau}{\sqrt{rr + \tau\tau}} = r - \frac{rr}{ſ} = r - \frac{r}{\sigma}\sqrt{\sigma\sigma - rr}.$

VIII. $V = 2r - v = r + \sqrt{rr - ss} = r + c = r + \frac{rr}{\sqrt{rr + tt}} = r + \frac{r\tau}{\sqrt{r^2 + \tau^2}} = r + \frac{r}{\sigma}\sqrt{\sigma^2 - rr} = \frac{s + t}{t}r = r + \frac{s\tau}{r} = r + \frac{rr}{ſ} = \frac{\tau + \sigma}{\sigma}r.$

FIG. 1.

The coverſed ſine is eaſily found expreſſed by any of the reſt, by only ſubtracting the ſine from radius.

PROP. II.

In any arch theſe lines are reſpectively proportional.

Radius : Sine of an arch : Coſine arch ::
2 Sine arch : Verſ. 2 arch : Sine 2 arch ::
2 Coſine arch : Sine 2 arch : Verſ. ſup. 2 arch.

For the triangles CAL, BAF, IBF are ſimilar, and IB = 2CL, therefore CA : AL : CL :: BA : AF : BF :: IB : BF : IF.

Cor. 1. From hence and Prop. 1 it is evident, that theſe are reſpectively proportional.

Radius : Sine of an arch : Coſine arch ::
Secant arch : Tangent arch : Radius ::
Coſecant arch : Radius : Cotangent arch ::
2 Sine arch : Verſ. 2 arch : Sine 2 arch ::
2 Coſine arch : Sine 2 arch : Verſ. ſup. 2 arch.

Cor. 2. *In any arch,* $\frac{2\ \text{ſine ſquare}}{\text{radius}}$ = *verſed ſine of the double arch, that is,* $\frac{2ss}{r}$ = *verſ.* 2A. See Schol. to Prop. 1.

Cor. 3. *In any arch,* $\frac{2\ \text{ſine} \times \text{coſine}}{\text{radius}}$ = *ſine of the double arch,* $\frac{2cs}{r}$ = ſin. 2A.

Cor. 4. *In any arch,* $\frac{2\ \text{coſine ſquare} - \text{radius ſquare}}{\text{radius}}$ = *coſine of the double arch.* For coſ. 2A = $r -$ verſ. 2A $= r - \frac{2ss}{r} = \frac{rr - 2ss}{r} = \frac{2cc - rr}{r}$.

Cor.

FIG. 1.

Cor. 5. *In any arch,* $\frac{2 \text{ radius ſquare}}{\text{cotangent} - \text{tangent}} =$ *tangent of twice the arch.* For coſ. : ſine :: rad. : tangent; that is, by Cor. 4, 3, $\frac{2cc - rr}{r}$ or $\frac{cc - ss}{r} : \frac{2cs}{r} :: r : \frac{2csrr}{cc - ss . r} = \frac{2rr}{\frac{rc}{s} - \frac{rs}{c}} =$ (by Schol. Pr. 1.) $\frac{2rr}{\tau - t}$.

Cor. 6. *In any arch,* $\frac{1}{2}$ *cotangent* — $\frac{1}{2}$ *tangent* = *cotangent of the double arch.* This follows from Cor. 5, becauſe radius ſquare divided by the tangent is = cotangent.

Cor. 7. *In any arch,* $\frac{\text{ſecant} \times \text{radius}}{2 \text{ coſine} - \text{ſecant}} =$ *ſecant of the double arch. For* $\frac{\text{radius ſquare}}{\text{coſine}} =$ *ſecant; that is, by Cor.* 4, $\frac{rr}{2cr - rr} r = \frac{\frac{rr}{c}}{2c - \frac{rr}{c}} r =$ (by Sch. Pr. 1.) $\frac{rſ}{2c - ſ} =$ ſecant of 2 arch.

Cor. 8. *In any arch,* $\frac{\text{ſecant} \times \text{coſecant}}{2 \text{ radius}} =$ *coſecant of the double arch.* For by Cor. 1 of this Prop. $\frac{2rs}{ſ} =$ ſin. 2 arch, and (by Sch. Pr. 1) $\frac{\text{radius ſquare}}{\text{ſine}}$ = coſecant; that is, $\frac{rrſ}{2rs} = \frac{ſ\sigma}{2r}$, becauſe $\frac{rr}{s} = \sigma$, by Schol. Pr. 1.

Cor. 9. *In any arch,* $\frac{8 \text{ ſine ſquare} \times \text{coſine ſquare}}{\text{radius cube}} =$ *verſed ſine of the quadruple arch.* For, by Cor. 3, $\frac{2cs}{r} =$ ſin. 2A, and by Cor. 2, $\frac{2 \times 4ccss}{r \times rr} =$ verſed ſine of twice 2A.

Cor.

FIG. 1. Cor. 10. *Hence alſo, the verſed ſines of two arches are as the ſquares of the ſines of their half arches.*

Cor. 11. *The ſines of two arches are as the rectangles of the ſine and coſine of their half-arches.*

Cor. 12. *The coſecants of two arches are as the rectangles of the ſecant and coſecant of their half arches.*

Cor. 13. *The verſed ſine of the ſup. of an arch : ½ radius :: ſine ſquare : ſine ſquare of ½ the arch.*

For verſ. ſup. 2A : coſ. A :: S.2A : S. A.
And 2 coſ. A : rad. :: S.2A : S. A. whence
by multiplying, verſ. ſup. 2A : ½rad. :: S^2.2A : S^2.A.

Cor. 14. *What has here been demonſtrated of the relation of ſines and coſines to one another, holds equally true, in reſpect of the cords of arches, and their ſupplements; putting the diameter inſtead of radius.*

Cor. 15. *Hence alſo it follows from Cor. 6, that the difference between the tangent and cotangent of an arch = twice the tangent of the difference between that arch and its complement.*

For the difference between an arch and its complement, is equal to the complement of twice that arch. Hence alſo

Cor. 16. *Let* A *be any arch leſs than* 45°, *then tangent of* $\overline{45+A}^{\circ}$ = *tangent of* $\overline{45-A}^{\circ}$ + 2 *tangents of* 2A. For 2A is the difference of the arches 45+A and 45−A.

SCHOLIUM.

Hence alſo the ſines, coſines, &c, of the double arches may be expreſſed more univerſally thus, (ſee Sch. Prop. 1.)

FIG. 1.

I. $\frac{2cs}{r} = \frac{2ss}{t} = \frac{2cc}{\tau} = \frac{2rs}{ſ} = \frac{2rc}{\sigma} = \frac{2rrt}{rr+tt=ſſ} = \frac{2rr}{\tau+t}$
$= \frac{2r^2\tau}{rr+\tau\tau=\sigma\sigma} =$ *ſine of* 2 *arch.*

II. $\frac{cc-ss}{r} = \frac{rr-2ss}{r} = \frac{2cc-rr}{r} = \frac{rr-tt}{rr+tt}r = \frac{\tau-t}{\tau+t}r$
$= \frac{\tau\tau-rr}{\tau\tau+rr}r = \frac{2rr-ſſ}{ſſ}r = \frac{2c-ſ}{ſ}r = \frac{\sigma^2-2r^2}{\sigma^2}r = \frac{\sigma-2s}{\sigma}r$
$=$ *coſine of* 2 *arch.*

III. $\frac{2rrt}{rr-tt} = \frac{2rr}{\tau-t} = \frac{2rcs}{rr-2ss} = \frac{2rcs}{2cc-rr} = \frac{2rr\tau}{\tau\tau-rr}$
$=$ *tangent of* 2 *arch.*

IV. $\frac{rr-tt}{2t} = \frac{\tau\tau-rr}{2\tau} = \frac{\tau-t}{2} = \frac{rr-2ss}{2cs}r = \frac{2cc-rr}{2cs}r$
$=$ *cotangent of* 2 *arch.*

V. $\frac{ſſr}{2rr-ſſ} = \frac{ſr}{2c-ſ} = \frac{r^3}{2cc-rr} = \frac{r^3}{rr-2ss} = \frac{rr+tt}{rr-tt}r$
$= \frac{\tau+t}{\tau-t}r = \frac{\tau\tau+rr}{\tau\tau-rr}r = \frac{r\sigma^2}{\sigma^2-2r^2} =$ *ſecant of* 2 *arch.*

VI. $\frac{ſ\sigma}{2r} = \frac{rſ}{2s} = \frac{r\sigma}{2c} = \frac{r^3}{2cs} = \frac{rr+tt=ſſ}{2t} = \frac{\tau+t}{2} =$
$\frac{rr+\tau\tau=\sigma\sigma}{2\tau} =$ *coſecant of* 2 *arch.*

VII. $\frac{2ss}{r} = \frac{2rr-2cc}{r} = \frac{2vV}{r} = \frac{2v\times\overline{2r-v}}{r} = \frac{2rtt}{rr+tt=ſſ}$
$= \frac{2rt}{\tau+t} = \frac{2r^3}{rr+\tau\tau=\sigma\sigma} = \frac{2rs}{\sigma} = \frac{ſſ-rr}{ſſ}\times 2r = \frac{ſ-c}{ſ}\times$
$2r =$ *verſed ſine of* 2 *arch.*

VIII. $\frac{2rr-2ss}{r} = \frac{2cc}{r} = \frac{2rr}{rr+tt}r = \frac{2\tau\tau}{rr+\tau\tau}r = \frac{2r\tau}{t+\tau}$

$=$

FIG.

$$=\frac{2r^3}{\text{ſſ}}=\frac{2tr}{\text{ſ}}=\frac{2\sigma^2-2r^2}{\sigma^2}r=\frac{2\sigma-2s}{\sigma}r=\frac{\overline{\mathrm{V}-r}|^2}{\frac{1}{2}r}=\frac{\overline{r-v}|^2}{\frac{1}{2}r}$$

= *verſ. ſup.* 2A.

The coverſed ſine may be expreſſed by any of the reſt, by ſubtracting the value of the ſine from radius. Alſo other values of each quantity may be inſerted by Sch. Prop. I.

Hence alſo,

I. $r\sqrt{\frac{r-c}{2r}}=r\sqrt{\frac{\text{ſ}-r}{2r}}=\sqrt{\frac{rv}{2}}=r\sqrt{\frac{2r-\mathrm{V}}{2r}}$
= S. ½ *the arch.*

II. $r\sqrt{\frac{r+c}{2r}}=r\sqrt{\frac{\text{ſ}+r}{2\text{ſ}}}=\sqrt{\frac{2rr-rv}{2}}=\sqrt{\frac{r\mathrm{V}}{2}}$
= *coſ.* ½ *the arch.*

III. $r\sqrt{\frac{r-c}{r+c}}=r\sqrt{\frac{\text{ſ}-r}{\text{ſ}+r}}=r\sqrt{\frac{v}{2r-v}}=r\sqrt{\frac{2r-\mathrm{V}}{\mathrm{V}}}$
$=\sigma-\tau=$ *tan.* ½ *the arch.*

IV. $r\sqrt{\frac{r+c}{r-c}}=r\sqrt{\frac{\text{ſ}+r}{\text{ſ}-r}}=\sqrt{\frac{2rr-rv}{2}}=r\sqrt{\frac{\mathrm{V}}{2r-\mathrm{V}}}$
$=\sigma+\tau=$ *cotan.* ½ *the arch.*

V. $r\sqrt{\frac{2r}{r+c}}=r\sqrt{\frac{2\text{ſ}}{r+\text{ſ}}}=r\sqrt{\frac{2r}{2r-v}}=\sqrt{\frac{2r^3}{\mathrm{V}}}$
= *ſec.* ½ *the arch.*

VI. $r\sqrt{\frac{2r}{r-c}}=r\sqrt{\frac{2\text{ſ}}{\text{ſ}-r}}=r\sqrt{\frac{2r}{v}}=r\sqrt{\frac{2r}{2r-\mathrm{V}}}$
= *coſec.* ½ *the arch.*

PROP. III.

2. *If three arches* AF, AN, AP *be in arithmetic progreſſion; then theſe are reſpectively proportional.*

Radius

FIG. 2.

Radius : *S. mean arch* : *Coſ. mean arch* ::
2 *S. mean arch* : *Verſ.* 2 *mean arch* : *S.* 2 *mean arch* ::
2 *Coſ. mean arch* : *S.* 2 *mean arch* : *Verſ. ſup.* 2 *mean arch* ::
2 *S. common diff.* : *Diff. coſ. extremes* : *Diff. ſin. extremes* ::
2 *Coſ. com. diff.* : *Sum. ſin. extremes* : *Sum. coſ. extremes* .

Note, *Diff. ſines = S. greater arch — S. leſſer arch.*
Diff. coſines = coſ. leſſer — coſ. greater arch =
Diff. verſed ſines = verſ. greater — verſ. leſſer arch, univerſally.

Draw PCM and PF and MF; and AD ∥ FP, and FS ∥ AB, and CE, MS ⊥ AB; Then MS = PG + FH, and SF = CG + CH. And arch OM = PN = NF. Therefore MF ∥ NC, and MF = 2CR. Whence the triangles CAQ, DAK, BDK, PFI, FMS are ſimilar; and CA : AQ : CQ :: DA : AK : DK :: BD : DK : BK :: PF : FI : PI :: FM : MS : FS.

Cor. 1. *Since the mean =* $\frac{1}{2}$ *ſum of the extremes, therefore in any two arches,*

Radius : *S.* $\frac{1}{2}$ *ſum* : *Coſ.* $\frac{1}{2}$ *ſum* ::
2 *S.* $\frac{1}{2}$ *ſum* : *Verſ. ſum* : *S. ſum* ::
2 *Coſ.* $\frac{1}{2}$ *ſum* : *S. ſum* : *Verſ. ſup. ſum* ::
2 *S.* $\frac{1}{2}$ *difference* : *Diff. coſines* : *Diff. ſines* ::
2 *Coſ.* $\frac{1}{2}$ *diff.* : *Sum ſines* : *Sum coſines.*

Cor. 2. *Let* A, E *be two arches,* A *the greater,* E *the leſſer, then*

$\frac{1}{2}$ *Rad.* : *S.* A. : *Coſ.* A. ::
S. E : *Coſ.* $\overline{A-E}$ — *Coſ.* $\overline{A+E}$: *S.* $\overline{A+E}$ — *S.* $\overline{A-E}$ ::
Coſ. E : *S.* $\overline{A+E}$ + *S.* $\overline{A-E}$: *Coſ.* $\overline{A-E}$ + *Coſ.* $\overline{A+E}$.

This will appear by Cor. 1, putting A = half the ſum of 2 arches, and E = half their difference. For then the arches will be $\overline{A+E}$, and $\overline{A-E}$.

Cor.

FIG. 2.

Cor. 3. *As ½ rad. : S. ½ ſum of two arches :: S. ½ difference : difference of their coſines, or of their verſed ſines.*

Cor. 4. *Rectangle of radius and the difference of the verſed ſines (or coſines) of two arches = twice the rectangle of the ſine of half the ſum, and ſine of half the difference of theſe arches.*

Cor. 5. *If three arches* AF, AN, AP *are in arithmetic progreſſion; the rectangle of the ſine of the mean arch, and coſine of the common difference = rectangle of radius, and half the ſum of the ſines of the extreme arches. And if three arches* AF, AN, AP *are in arithmetic progreſſion: rectangle of the coſine of the mean arch, and coſine of the common difference = rectangle of radius and half the ſum of the coſines of the extremes.*

For by Cor. 1, $CA \times \overline{PG + FH} = 2CR \times AQ = 2AQ \times \overline{CN - NR}$, whence $2AQ \times NR = 2AQ \times CA - CA \times \overline{PG + FH}$. And $\overline{CA - NR} \times AQ$ or $CR \times AQ = CA \times \frac{PG + FH}{2}$. And if EF, EN, EP be the arches, AQ, PG, FH are coſines.

Cor. 6. *The ſecond difference of the ſines* FH, AQ, PG, × *radius = twice the ſine of the mean arch × verſed ſine of the common difference.* For the ſecond difference $= 2AQ - FH - PG$.

Cor. 7. *As radius:*

To twice the coſine of any arch ::
So the ſine of n times that arch :
To ſum of the ſines of n — 1 and n + 1 times that arch ::
And ſo the coſine of n times that arch :
To the ſum of the coſines of n — 1 and n + 1 times that arch.

For if A be the arch, then $\overline{n-1}.A$, nA, $\overline{n+1}.A$ are in arithmetic progreſſion, and A the common difference. FIG. 2.

Cor. 8. *Let* A *be any arch leſs than* 30°, *then ſine of* $A\times\sqrt{3}+$ *ſine of* $\overline{30-A}=$ *ſine of* $\overline{30+A}$.

For the arches $30-A$, 30°, and $30+A$ are in arithmetic progreſſion, and coſ. mean $30=\frac{1}{2}\sqrt{3}\times$ radius. And by this prop. rad. : coſ. 30° :: 2 ſine of A : S. $\overline{30+A}-$S. $\overline{30-A}$.

Cor. 9. *If* A *be any arch leſs than* 30, *then* S. $\overline{30+A}+$S. $\overline{30-A}=$ S. $\overline{90-A}$.

For the S. 30 the mean $=\frac{1}{2}$ rad. and by this prop. rad. : $\frac{1}{2}$ rad. :: 2 coſ. A : S. $\overline{30+A}+$S. $\overline{30-A}=$ coſ. A.

Cor. 10. *If* A *be any arch leſs than* 30°, *then* S. $\overline{60+A}\times\sqrt{3}-$S. $\overline{30+A}=$ S. $\overline{90-A}$.

For if the arches be A, $60+A$, $120+A$, and radius $\times\frac{1}{2}\sqrt{3}=$ S. 60 the common difference. Then rad. : rad. $\times\sqrt{3}$ :: S. $\overline{60+A}$: coſ. $A-$coſ. $\overline{120+A}$ $=$ coſ. $A+$coſ. $\overline{60-A}$, that is, 1 : $\sqrt{3}$ :: S. $\overline{60+A}$: S. $\overline{90-A}+$S. $\overline{30+A}$.

Cor. 11. *If* A *be any arch leſs than* 45°; *then S.* $A\times\sqrt{2}+$*S.* $\overline{45-A}=$*S.* $\overline{45\pm A}$. For in the arches $45-A$, 45, $45+A$; $\frac{1}{2}\sqrt{2}\times$ rad. $=$ coſ. the mean, 45.

Cor. 12. *Let* A *be any arch leſs than* 60; *then S.* $A+$ *S.* $\overline{60-A}=$ *S.* $\overline{60+A}$. For in the arches $60-A$, 60, $60+A$, the coſ. mean, 60° $=\frac{1}{2}$ rad.

Cor. 13. *What is demonſtrated in this prop. and Cor.* 2, 3, 5, 7, *of ſines and coſines, holds in cords of arches,* FIG. 2. *arches, and their ſupplements, putting diameter for radius.*

PROP. IV.

In any two arches; theſe are reſpectively proportional.

$\frac{1}{2}$ *radius* : *S.* $\frac{1}{2}$ *ſum* : *Coſ.* $\frac{1}{2}$ *ſum* : *S.* $\frac{1}{2}$ *diff.* : *Coſ.* $\frac{1}{2}$ *diff.* : :
S. $\frac{1}{2}$ *ſum* : *Verſ. ſum* : *S. ſum* : *Diff. coſ.* : *Sum ſines* : :
Coſ. $\frac{1}{2}$ *ſum* : *S. ſum* : *V. ſup. ſum* : *Diff. ſines* : *Sum coſ.* : :
S. $\frac{1}{2}$ *diff.* : *Diff. coſ.* : *Diff. ſines* : *Verſ. diff.* : *S. diff.* : :
Coſ. $\frac{1}{2}$ *diff.* : *Sum ſines* : *Sum coſ.* : *S. diff.* : *V. ſup. diff.*

Note, *Diff. ſines and coſines, is the ſame as in Pr.* III.

The 3 firſt perpendicular rows are evident from the laſt Prop. and the firſt, fourth, and fifth terms in the 2 laſt perpendicular rows follow from Prop. II, and the remaining terms are filled up by the proportions in this very prop. thus, $\frac{1}{2}$ rad. : S. $\frac{1}{2}$ diff. : : S. $\frac{1}{2}$ ſum : diff. coſines, which therefore will ſtand in the ſecond place of the fourth perpendicular column, as well as in the fourth place of the ſecond column, &c.

Hence ſeveral corollaries follow of their own accord; as,

Cor. 1. *As verſ. ſuppl. ſum of 2 arches : diff. ſines : : diff. ſines : verſed ſine of their difference.*

Cor. 2. *Sine of the ſum of 2 arches : ſum of their ſines : : difference of their ſines : ſine of the difference of the arches.*

Cor. 3. *If three arches are in arithmetical progreſſion,*
As the ſine of one extreme :
Sum of the ſines of the mean arch, and common difference : :
So their difference (S. mean—S. com. diff.)
To the ſine of the other extreme.

For

FIG. 2.

For let m = mean arch, d = com. difference, and then by Cor. 2. S. $\overline{m+d}$: S. m+S. d :: S. m−S. d : S. $\overline{m-d}$.

Cor. 4. Let A be any arch; m, n any numbers, then
As ſine of $\overline{m-n}$. A :
S. mA+*S.* nA ::
S. mA−*S.* nA :
Sine of $\overline{m+n}$. A. By this prop. Hence

Cor. 5. *As the ſine of an arch :*
Sum of the ſines of the double and ſingle arch ::
So their difference (S. double arch — S. ſingle arch) :
To ſine of the triple arch :

Cor. 6. *As the ſine of an arch :*
Sum of the ſines of triple and double ::
So their difference (S. triple — S. double) :
To ſine of the quintuple arch, &c.

Cor. 7. *Let* r = *radius,* s = *ſine of the arch* 2A; *then the ſine of* $\overline{45+A} = \sqrt{\frac{rr+rs}{2}}$, *and the ſine of* $\overline{45-A} = \sqrt{\frac{rr-rs}{2}}$.

For let a = S. $\overline{45+A}$, e = S. $\overline{45-A}$. Then, by this prop. as (S. ſum) r : (ſum ſines) $a+e$ :: (diff. ſines) $a-e$: s the ſine of the difference. Therefore,

$$rs = aa - ee$$
but $rr = aa + ee$ by Cor. 1 Prop. 1;
therefore $rr + rs = 2aa$
and $rr - rs = 2ee$.

Cor. 8. *What has been demonſtrated in this prop. and firſt 6 corollaries, concerning the relation of ſines and coſines, holds equally true with reſpect to their cords; putting the cord of the ſupplement inſtead of coſine, and the diameter inſtead of radius.*

SCHOLIUM.

Let there be two arches A, E; and let

$$\text{C} = \text{coſ.}\ \overline{\text{A}+\text{E}},\quad c = \text{coſ.}\ \overline{\text{A}-\text{E}},$$

$$\text{V} = \text{verſ.}\ \overline{\text{A}+\text{E}},\quad v = \text{verſ.}\ \overline{\text{A}-\text{E}},$$

$$\text{Z} = \text{ſine}\ \frac{\text{A}+\text{E}}{2},\quad z = \text{S.}\frac{\text{A}-\text{E}}{2},$$

$$\text{Y} = \text{coſ.}\frac{\text{A}+\text{E}}{2},\quad y = \text{coſ.}\frac{\text{A}-\text{E}}{2},$$

$$\text{D} = \text{coſ.}\ \overline{2\text{A}+2\text{E}},\quad d = \text{coſ.}\ \overline{2\text{A}-2\text{E}}.$$

Then

$$\text{S.A}\times\text{S.E} = \overline{\text{Z}+z}\times\overline{\text{Z}-z} = \text{Z}^2-z^2 = \overline{y+\text{Y}}\times\overline{y-\text{Y}}$$

$$= y^2-\text{Y}^2 = \frac{\text{V}-v}{2}r = \frac{c-\text{C}}{2}r;$$

$$\text{Coſ. A}\times\text{coſ. E} = \frac{c+\text{C}}{2}r;$$

$$\text{Tan. A}\times\text{tan. E} = \frac{c-\text{C}}{c+\text{C}}rr;$$

$$\text{S. A}\times\text{S. E}\times\text{coſ. A}\times\text{coſ. E} = \frac{cc-\text{CC}}{4}rr = \frac{d-\text{D}}{8}r^3;$$

$$\text{Coſ. A}\times\text{coſ. E}\times\text{tan. A}\times\text{tan. E} = \frac{c-\text{C}}{2}r^3.$$

PROP. V.

3. *In any two arches* (AD, DE), *the ſum of the rectangles of the ſine of one into the coſine of the other, is* = *rectangle of the radius and the ſine of the ſum of theſe arches.*

DEMONSTRATION.

Draw NF ∥ and NR ⊥ CA: then, becauſe the angles ENC, FNR are right, therefore ENF = CNR; and therefore the triangles ENF, CNR and CDG are ſimilar; wherefore CD : DG :: CN : NR $= \frac{\text{DG}\times\text{CN}}{\text{CD}}$ = FI. And CD : CG :: EN : EF $= \frac{\text{CG}\times\text{EN}}{\text{CD}}$, whence EI = EF + FI $= \frac{\text{CG}\times\text{EN}+\text{DG}\times\text{CN}}{\text{CD}}$. Q. E. D.

Cor.

FIG. 3.

Cor. I. *In any two arches* (AD, DE), *the rectangle of their coſines — the rectangle of their ſines = the rectangle of radius and the coſine of their ſum.*

For by ſimilar triangles, $\mathrm{CD} : \mathrm{CG} :: \mathrm{CN} : \mathrm{CR} = \frac{\mathrm{CG} \times \mathrm{CN}}{\mathrm{CD}}$. And $\mathrm{CD} : \mathrm{DG} :: (\mathrm{CN} : \mathrm{NR} ::)\ \mathrm{EN} : \mathrm{NF} = \frac{\mathrm{DG} \times \mathrm{EN}}{\mathrm{CD}} = \mathrm{IR}$. And $\mathrm{CI} = \mathrm{CR} - \mathrm{IR} = \frac{\mathrm{CG} \times \mathrm{CN} - \mathrm{DG} \times \mathrm{EN}}{\mathrm{CD}}$.

Cor. 2. *The ſquare of the ſine of the ſum of two arches is = the ſum of the ſquares of their ſines + twice their rectangle, multiplied by the coſine of the ſum and divided by radius.*

For let the arches $\mathrm{A} + \mathrm{E} = \mathrm{S}$, their ſines $a + e = s$; and c the coſine of S. Then by this prop. $s = \frac{a\sqrt{rr - ee} + e\sqrt{ra -- aa}}{r}$, and $ss = aa + ee - \frac{2adee}{rr} + \frac{2ae}{rr}\sqrt{rr - aa} \times \sqrt{rr - ee}$. But by Cor. I. $\frac{\sqrt{rr - aa} \times \sqrt{rr - ee}}{r} - \frac{ae}{r} = c$. Therefore $ss = aa + ee + \frac{2c}{r}ae$.

Cor. 3. *Hence if* $\mathrm{A} + \mathrm{E} = 90°$, *then* $aa + ee = rr$.
If $\mathrm{A} + \mathrm{E} = 60$, *then* $aa + ee + ae = \frac{3}{4}rr$.
If $\mathrm{A} + \mathrm{E} = 45$, *then* $aa + ee + ae\sqrt{2} = \frac{1}{2}rr$.
If $\mathrm{A} + \mathrm{E} = 30$, *then* $aa + ee + ae\sqrt{3} = \frac{1}{4}rr$.

SCHOLIUM.

This Prop. and Cor. I. hold univerſally for all arches, putting negative coſines for arches greater than a quadrant.

PROP. VI.

In two arches (AD, AE), *the rectangle of the ſine of the greater and coſine of the leſſer — the rectangle of the*

FIG. 3. *the ſine of the leſſer and coſine of the greater, is* = *rectangle of radius and the ſine of the difference of the arches.*

DEMONSTRATION.

The triangles CDG, CBI, and EBN, are ſimilar; therefore CG : DG :: CI : BI = $\frac{CI \times DG}{CG}$.

And CD : CG :: EB or EI $- \frac{CI \times DG}{CG}$: EN = $\frac{CG \times EI - CI \times DG}{CD}$.

Cor. *In any two arches* (AD, AE), *the rectangle of their ſines* + *the rectangle of their coſines, is* = *rectangle of the radius and the coſine of the difference of the arches.*

For BI = $\frac{CI \times DG}{CG}$, and by ſimilar triangles CB = $\frac{CI \times CD}{CG}$. And CD : DG :: BE or EI $- \frac{DG \times CI}{CG}$: BN = $\frac{DG \times EI}{CD} - \frac{DG^2 \times CI}{CD \times CG} = \frac{DG \times EI}{CD} - \frac{CD^2 \times CI}{CD \times CG} + \frac{CG^2 \times CI}{CD \times CG} = \frac{DG \times EI}{CD} - CB + \frac{CG \times CI}{CD}$; therefore $\frac{DG \times EI + CG \times CI}{CD}$ = CB + BN = CN.

SCHOLIUM I.

This prop. and its cor. hold univerſally for all arches, putting negative coſines for arches greater than a quadrant.

SCHOLIUM II.

From the foregoing propoſitions it follows, that if radius = 1. And

s, c = ſine, and coſine of half the ſum of two arches.

s, c = ſine, and coſine of half their difference; then FIG. 3.

$\mathrm{S}c + \mathrm{C}s$ = ſine of the greater;
$\mathrm{S}c - \mathrm{C}s$ = ſine of the leſſer;
$\mathrm{C}c - \mathrm{S}s$ = coſine of the greater;
$\mathrm{C}c + \mathrm{S}s$ = coſine of the leſſer.

Alſo

If Z = ſecant of the greater arch, z = ſecant of the leſſer;
Y = coſecant of the greater, y = coſecant of the leſſer.

Then $\dfrac{\mathrm{Y}y\mathrm{Z}z}{\mathrm{Y}y - \mathrm{Z}z}$ = ſecant of the ſum;

$\dfrac{\mathrm{Y}y\mathrm{Z}z}{\mathrm{Z}y + \mathrm{Y}z}$ = coſecant of the ſum;

$\dfrac{\mathrm{Y}y\mathrm{Z}z}{\mathrm{Y}y + \mathrm{Z}z}$ = ſecant of the difference;

$\dfrac{\mathrm{Y}y\mathrm{Z}z}{\mathrm{Z}y - \mathrm{Y}z}$ = coſecant of the difference.

PROP. VII.

If three arches (AF, AN, AP), *be in arithmetic progreſſion,* 4.
The ſum of the ſines of the extreme arches :
To their difference (S. greater — S. leſſer) ::
As tang. mean arch :
To tang. common difference of the arches.

For drawing the lines as in the figure, then by ſimilar triangles, $2\mathrm{OB}$ or $\mathrm{PG} + \mathrm{FH} : 2\mathrm{OD}$ or $\mathrm{PI} :: \mathrm{OB} : \mathrm{OD} :: \mathrm{OL} : \mathrm{OF} :: \mathrm{NM} : \mathrm{NK}$.

Cor. 1. In any two arches (AF, AP;)
As ſum of their ſines :
To their difference (S. greater — S. leſſer) ::
Tang. half their ſum :
Tang. half their difference.

For the mean arch = $\frac{1}{2}$ ſum of the extremes.

FIG. 4. *Cor.* 2. In any two arches (AF, AP),

The ſum of their coſines :
Their diff. (coſ. leſſer — coſ. greater) : :
As cotang. half their ſum :
Tang. half their difference.

For CH + CG or 2CB : CH — CG or 2DF : : CB : BH : : RO : OF : : SN : NK.

Cor. 3. In any two arches (AF, AP), *as*
S. greater : *S. leſſer* : : *T.* $\frac{1}{2}$ *ſum* + *T.* $\frac{1}{2}$ *diff.* : *T.* $\frac{1}{2}$ *ſum* — *T.* $\frac{1}{2}$ *diff.*

For by the demonſtration of this prop. PG + FH : PG — FH : : MN : NK. And by compoſition and diviſion, PG : FH : : QM : KM.

Cor. 4. In any two arches (AN, NP;) *as*

The ſum of their tangents :
Their diff. (tan. greater — tan. leſſer) : :
Sine of their ſum :
Sine of their difference.

Let AN be the greater, and make NF = NP; then by ſimilar triangles, QM : KM : : (PL : FL : :) PG : FH.

Cor. 5. In any two arches (AN, NP;) *as*
Tan. greater : *tan. leſſer* : : *S. ſum* + *S. difference* : *S. ſum* — *S. difference.*

For let AN be the greater, and make NF = NP; then AP is the ſum, and AF is the difference of the arches; and by Cor. 4 we have QM : KM : : PG : FH. And by compoſition and diviſion,

$$\frac{QM + KM}{2} \text{ or } NM : \frac{QM - KM}{2} \text{ or } NQ :: PG + FH : PG - FH.$$

Cor. 6. *Let* A *be any arch leſs than* 45°; *then*

S. $\overline{45 + A}$ + *coſ.* $\overline{45 + A}$:
S. $\overline{45 + A}$ — *coſ.* $\overline{45 + A}$: :
Radius :
Tan. A.

This appears by this prop. putting $AN = 45°$, and $NF = NP = A$. FIG. 4.

SCHOLIUM.

The propoſitions here delivered hold univerſally, obſerving to take negative coſines and tangents for arches greater than a quadrant.

PROP. VIII.

In any two arches, AE, ED ; 5.
Radius ſquare — rectangle of their tangents :
Radius ſquare : :
Sum of their tangents :
Tangent of the Sum of the arches.

For by ſimilar triangles CB or $CA - BA : CK$ or $CF - KF :: CF : CA$. Whence $CA^2 - BAC = CF^2 - KFC$, and $KFC = CF^2 - CA^2 + BAC = FA^2 + BAC$. Again, $CB : BK :: CF : FA$, and $BK : DG :: KF : FD$ or FA, therefore *ex equo* CB or $CA - AB : DG :: CFK$ or $FA^2 + BAC : AF^2$; therefore $\overline{AC - AB} \times AF^2 = DG \times FA^2 + DG \times BAC$. And $\overline{AB + DG} \times AF^2 = AC \times \overline{AF^2 - AB \times DG}$. Q. E. D.

Cor. I. *Let* T, t *be the tangents*; X, x, *the cotangents of two arches*; *then* $\frac{rr - Tt}{T + t} = \frac{Xx - rr}{X + x} =$ *cotangent of the ſum of the arches.*

For let $r =$ radius, $\tau =$ cotangent of the ſum. Then $AC = \frac{rr}{\tau}$, and $\overline{AB + DG} \times rr = \frac{rr}{\tau} \times \overline{rr - AB \times DG}$; whence $\tau = \frac{rr - AB \times DG}{AB + DG} = \frac{\textit{rect. cot.} - rr}{\textit{ſum. cot.}}$.

FIG. 5. Cor. 2. *If one of the arches be 45°, t = tang. other arch; then* $\frac{r+t}{r-t}\times r$ = *tang. ſum of the arches.*

SCHOLIUM.

What belongs in general to the addition of tangents and cotangents, is delivered in this prop. and its cors. All caſes where the arches are greater than a quadrant will eaſily appear by putting negative tangents and cotangents for theſe arches.

PROP. IX.

In any two arches AD, AE; *as*
Radius ſquare + rectangle of their tangents :
Radius ſquare ::
Difference (or tangent greater — tangent leſſer) :
Tangent of the difference of the arches.

For you will find as in the laſt prop. CB : DG :: CFK or FA^2 + BAC : FA^2, Q. E. D.

Cor. 1. *Let* T, t *be the tangents,* X, x *the cotangents of two arches; then* $\frac{rr+Tt}{T-t}=\frac{rr+Xx}{x-X}$ = *cotangent of the difference of the arches.*

This appears by putting $\frac{rr}{T}$ for the tangent of the difference.

Cor. 2. *If one of the arches be 45°, t = tangent of the other, then* $\frac{r-t}{r+t}r$ = *tangent of their difference* = *tan.* $\overline{45^\circ - \text{the other arch}}$, *when the other arch is leſſer than 45°; or* $\frac{t-r}{t+r}r$ = *tan.* $\overline{\text{the other arch} - 45^\circ}$, *when greater.*

SCHO-

FIG. 5.

SCHOLIUM.

For arches greater than a quadrant, put negative tangents and cotangents.

PROP. X.

The ſecant of an arch is equal to the ſum of the tangent of it and the tangent of half its complement.

For let A = arch, T its tangent, $ſ$ its ſecant; a = ½ complement, t its tangent. Then by Schol. Pr. II, $ſ=\frac{rr+tt}{2t}$, and $T=\frac{rr-tt}{2t}$, and $ſ-T=t$; whence ſecant of A = tangent of A + tang. ½ complement of A.

Cor. 1. *The ſecant of an arch = cotangent of half the complement — the tangent of the arch.*

For $ſ+T=\frac{rr}{t}$ = cot. a = cotang. ½ compl. of A.

Cor. 2. *Half the ſum of the tangent and cotangent of an arch is = ſecant of the difference between that arch and its complement.*

For by Schol. Pr. II, ½ tangent + ½ cotangent = coſecant of the double arch; and the complement of the double arch is the ſame as the difference between the arch and its complement.

Cor. 3. *As radius + ſecant : radius :: tangent : tangent of half the arch.* For $r+\frac{T+t}{T-t}r$ or $\frac{2rT}{T-t}:r::\frac{2rr}{T-t}:t$. See Sch. Pr. II.

Cor. 4. *Secant of an arch* A = *tangent of* $\overline{45+\frac{1}{2}A}^{\circ}$ — *tangent of* A. This follows from Cor. 1.

SCHOLIUM.

From what has been before laid down, it will not be difficult to find the ſines of as many arches as we will, FIG. 5. will, expreſſed in ſurd numbers. As in theſe following, where the prop. in the margin ſhows whence and how they are derived, either immediately, or by help of what ſtep.

Pr. IV, 7, Step 8.	1	S. 15°	$= \frac{1}{2}r \sqrt{2-\sqrt{3}}$.
Pr. IV, 7, and II, 3.	2	S. 18	$= r \times \frac{\sqrt{5}-1}{4}$.
Pr. IV, 7, Step 6.	3	S. 22 $\frac{1}{2}$	$= \frac{1}{2}r \sqrt{2-\sqrt{2}}$.
Pr. III, Step 12.	4	S. 30	$= \frac{1}{2}r$.
Pr. IV, 7, Step 2.	5	S. 36	$= \frac{1}{4}r \sqrt{10-2\sqrt{5}}$.
Pr. IV, 7,	6	S. 45	$= \frac{1}{2}r \sqrt{2}$.
Pr. IV, 7, Step 2.	7	S. 54	$= r \times \frac{\sqrt{5}+1}{4}$.
Pr. IV, 7, Step. 4.	8	S. 60	$= \frac{1}{2}r \sqrt{3}$.
Pr. IV, 7, Step. 6.	9	S. 67 $\frac{1}{2}$	$= \frac{1}{2}r \sqrt{2+\sqrt{2}}$.
Pr. I, 1, Step 2.	10	S. 72	$= \frac{1}{4}r \sqrt{10+2\sqrt{5}}$.
Pr. IV, 7, Step. 8.	11	S. 75	$= \frac{1}{2}r \sqrt{2+\sqrt{3}}$.
Def. 2.	12	S. 90	$= r$.

And thus you may find as many ſines as you pleaſe by Prop. III, Cor. 6, 7, 8, 9. And Prop. IV, Cor. 7, and Prop. V, Cor. 4, 5, by the help of theſe already found; but then they will be ſtill more and more compounded with ſurd quantities, except you chuſe to extract them. The ſine of 18 in ſtep the ſecond is not ſo eaſily derived as the reſt; it is had thus. Put $s =$ S. 18. Then by Pr. IV, 7, S. 36 $= \sqrt{\frac{rr-rs}{2}}$, and by Prop. II, 3, the ſame ſine of 36 $= \frac{2s}{r}\sqrt{rr-ss}$; which equated and divided by $\sqrt{r-s}$, and then reduced, you have a cubic equation, $8s^3 + 8rss = r^3$. And the root is $s = \frac{\sqrt{5}-1}{4} \times r$.

SECT. II.

The calculation of natural ſines, tangents, and ſecants of arches.

PROP. XI.

The radius and ſine AB *of an arch* DB *being given; to find the arch.* FIG. 6.

Take the arch Bb infinitely ſmall, and draw Bs ∥ CD, and bsu ∥ BA; and let CB or CD $= r$, AB $= y$, DB $= z$, CA $= \sqrt{rr-yy}$, B$b = \dot{z}$, $sb = \dot{y}$. The triangles CAB, and Bsb are ſimilar: therefore CA $(\sqrt{rr-yy})$: CB (r) : : sb $(\dot{y})$: Bb or $\dot{z} = \frac{r\dot{y}}{\sqrt{rr-yy}}$. And the fluent is $z = y + \frac{y^3}{3.2r^2} + \frac{3.y^5}{5.2.4r^4} + \frac{3.5y^7}{7.2.4.6r^6} + \frac{3.5.7y^9}{9.2.4.6.8.r^8}$ &c, or arch DB $= y + \frac{yy}{2.3rr}\text{A} + \frac{3.3y^2}{4.5rr}\text{B} + \frac{5.5y^2}{6.7r^2}\text{C} + \frac{7.7y^2}{8.9r^2}\text{D}$, &c; putting A, B, C, &c, for the firſt, ſecond, third terms, &c.

Cor. 1. *If $d =$ diameter, $c =$ cord of an arch; then the arch* $= c + \frac{cc}{2.3d^2}\text{A} + \frac{3^2c^2}{4.5d^2}\text{B} + \frac{5^2c^2}{6.7d^2}\text{C}$, &c.

Cor. 2. *Let* $Q =$ *a quadrant; then the arch whoſe coſine is y is* $= Q - y - \frac{y^3}{2.3r^2} - \frac{3y^5}{2.4.5r^4} - \frac{3.5y^7}{2.4.6.7r^6}$ *&c.*

Or the arch whoſe coſine is y is $= \frac{r-y}{1} + \frac{r^3-y^3}{2.3r^2} + 3 \times \frac{r^5-y^5}{2.4.5r^4} + 3 \times 5 \times \frac{r^7-y^7}{2.4.6.7r^6}$ *&c.*

PROP.

6. FIG.

PROP. XII.

The arch BD *being given; to find the ſine* BA.

Let CD $= r$, AB $= y$, arch DB $= z$. Then by Prop. XI, $\dot z = \frac{r\dot y}{\sqrt{rr-yy}}$, and $\overline{rr-yy} \times \dot z^2 = rr\dot y^2$: this put into fluxions, making $\dot z$ conſtant, we have $-2y\dot y\dot z^2 = 2rr\dot y\ddot y$, or $rr\ddot y = -y\dot z^2$. Now when z is infinitely ſmall, $z = y$; therefore aſſume,

$$y = z + bz^3 + cz^5 + dz^7 \text{ \&c,}$$

$$\text{then } \dot y = \dot z + 3bz^2\dot z + 5cz^4\dot z + 7dz^6\dot z \text{ \&c,}$$

$$\text{and } \ddot y = 2.3bz\dot z^2 + 4.5cz^3\dot z^2 + 6.7dz^5\dot z^2 \text{ \&c,}$$

then $rr\ddot y = -y\dot z^2$; that is,

$$\begin{aligned} &2.3brrz\dot z^2 + 4.5crrz^3\dot z^2 + 6.7rrdz^5\dot z^2 + \text{\&c,} \\ &= -z\dot z^2 - bz^3\dot z^2 - cz^5\dot z^2 - \text{\&c.} \end{aligned}$$

Therefore, equating the coefficients, $b = \frac{-1}{2.3rr}$, $c = \frac{-b}{4.5rr}$, $d = \frac{-c}{6.7rr}$, $e = \frac{-d}{8.9rr}$, &c; and $y = z - \frac{1}{2.3rr}z^3 - \frac{b}{4.5rr}z^5$ &c, that is, $y = z - \frac{z^3}{2.3rr} + \frac{z^5}{2.3.4.5r^4} - \frac{z^7}{2.3.4.5.6.7r^6} + \text{\&c}$; that is, y or ſine AB $= z - \frac{zz}{2.3rr}\text{A} - \frac{zz}{4.5rr}\text{B} - \frac{zz}{6.7rr}\text{C} - \frac{zz}{8.9rr}\text{D}$, &c; where A, B, C, &c, are the foregoing terms with their ſigns.

Cor. 1. *If* z *be any arch, its coſine* CA $= r - \frac{zz}{1.2rr}\text{A} - \frac{zz}{3.4rr}\text{B} - \frac{zz}{5.6rr}\text{C} - \frac{zz}{7.8rr}\text{D} - \text{\&c.}$

For CA $= \sqrt{rr-yy} = \sqrt{rr - zz + \frac{z^4}{3rr} - \frac{2z^6}{45r^4} + \frac{z^8}{315r^6} - \text{\&c}} = r - \frac{zz}{1.2r} + \frac{z^4}{1.2.3.4r^3} - \frac{z^6}{1\ldots.6r^5} + \frac{z^8}{1\ldots.8r^7} - \text{\&c.}$

Cor. 2. *If d be the diameter, z any arch; then the* cord $= z - \frac{zz}{2.3dd}A - \frac{zz}{4.5dd}B - \frac{zz}{6.7dd}C - \frac{zz}{8.9dd}D$ — &c. FIG. 6.

PROP. XIII.

The ſine and coſine of an arch A *being given; and if there be given another arch* z; *to find the ſine and coſine of the ſum of the arches* A + z, *and alſo of the difference* A — z.

Let s, c be the ſine and coſine of A. And by Prop. XII and Cor. 1, the ſine of the arch z is $z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4}$ &c $=$ B. And its coſine $r - \frac{z^2}{2r} + \frac{z^4}{24r^3}$ &c $=$ D. Then by Prop. V and Cor. 1, and Prop. VI and Cor. $\frac{sD + cB}{r} = S.\overline{A + z}$. And $\frac{cD - sB}{r} =$ coſ. $\overline{A + z}$. And $\frac{sD - cB}{r} = S.\overline{A - z}$. And $\frac{sB + cD}{r} =$ coſ. $\overline{A - z}$; that is,

$$S.\overline{A + z} = s + \frac{cz}{r} - \frac{sz^2}{1.2r^2} - \frac{cz^3}{2.3r^3} + \frac{sz^4}{1.2.3.4r^4} + \frac{cz^5}{1.2.3.4.5r^5} - \&c.$$

$$\text{Coſ. } \overline{A + z} = c - \frac{sz}{r} - \frac{cz^2}{2r^2} + \frac{sz^3}{2.3r^3} + \frac{cz^4}{2.3.4r^4} - \frac{sz^5}{2.3.4.5r^5} - \&c.$$

$$S.\overline{A - z} = s - \frac{cz}{r} - \frac{sz^2}{2r^2} + \frac{cz^3}{2.3r^3} + \frac{sz^4}{2.3.4r^4} - \frac{cz^5}{2.3.4.5r^5} - \&c.$$

Coſ.

FIG.

Coſ. $\overline{A-z} = c + \frac{sz}{r} - \frac{cz^2}{2r^2} - \frac{sz^3}{2.3r^3} + \frac{cz^4}{2.3.4r^4} + \frac{sz^5}{2.3.4.5r^5} -$ &c.

Cor. 1. *If there be given s, c, the ſine and coſine of an arch* A; *and* $s + x$ *be the ſine of another arch* $\overline{A+z}$; *then the difference of the ſines* $x = \frac{cz}{r} - \frac{sz^2}{2r^2} - \frac{cz^3}{2.3r^3} + \frac{sz^4}{2.3.4r^4}$. *&c.*

And the difference of the arches $z = \frac{r}{c} \times : x + \frac{sx^2}{2c^2} + \frac{rr+2ss}{6c^4}x^3 + \frac{3r^2+2ss}{8c^6}sx^4$ *&c.*

For by this prop. the ſine of $\overline{A+z}$, or $s + x = s + \frac{cz}{r} - \frac{sz^2}{2rr}$ &c, and $x = \frac{cz}{r} - \frac{sz^2}{2rr}$ &c, and by reverſion of ſeries, z is found as above.

Cor. 2. *The ſecond difference of the ſines of* $A - z$, A, $A + z$, *is* $= 2s \times : \frac{z^2}{1.2r^2} - \frac{z^4}{1.2.3.4r^4} + \frac{z^6}{1.2.3.4.5.6r^6} -$ *&c*; (for in any three things, the ſecond difference is equal to the difference between twice the mean, and the ſum of the extremes.)

PROP. XIV.

6. *The verſed ſine of an arch* DB *being given; to find the arch.*

Let $CD = r$, $DA = v$, $Au = \dot{v}$, $DB = z$. Then $AB = \sqrt{2rv - vv}$. The triangles CAB and B*sb* are ſimilar, therefore $\sqrt{2rv-vv} : r :: \dot{v} : Bb$ or $\dot{z} = \frac{r\dot{v}}{\sqrt{2rv-vv}}$. And the fluent is $z = \sqrt{2rv} \times : 1 + \frac{v}{3.2^2r}$

$\frac{v}{3.2^2r} + \frac{3v^2}{2^3.4.5r^2} + \frac{3.5v^3}{2^4.4.6.7r^3}$, &c. that is, arch DB $= \sqrt{2rv} + \frac{v}{3.4r}\mathrm{A} + \frac{3^2v}{5.8r}\mathrm{B} + \frac{5^2v}{7.12r}\mathrm{C} + \frac{7^2v}{9.16r}\mathrm{D}$ &c. Or rather DB $= \sqrt{dv} + \frac{v}{2.3d}\mathrm{A} + \frac{3^2v}{4.5d}\mathrm{B} + \frac{5^2v}{6.7d}\mathrm{C}$ &c, putting $d =$ diameter; where A, B, C are the foregoing terms.

Cor. 1. *In any arch a, the verſed ſine is* $= \frac{aa}{2r} - \frac{aa}{3.4r^2}\mathrm{A} - \frac{aa}{5.6r^2}\mathrm{B} - \frac{aa}{7.8r^2}\mathrm{C} -$ &c, *where* A, B, C *are the foregoing terms with their ſigns.*

For radius — coſine is equal to the verſed ſine, and by Cor. 1 Pr. XII, the coſine is $r - \frac{a^2}{1.2r} + \frac{a^4}{1.2.3.4r^3}$ &c.

Cor. 2. *The coverſed ſine of the arch a, is* $= r - a + \frac{a^3}{2.3r^2} - \frac{a^5}{2.3.4.5r^4} + \frac{a^7}{2.3.4.5.6.7r^6}$ &c; for it is $=$ radius — ſine.

Cor. 3. *Verſed ſine of the ſupplement is* $= 2r - \frac{aa}{1.2r^2}\mathrm{A} - \frac{aa}{3.4r^2}\mathrm{B} - \frac{aa}{5.6r^2}\mathrm{C} - \frac{aa}{7.8r^2}\mathrm{D}$, &c; for it is $=$ radius $+$ coſine.

PROP. XV.

The tangent DT *of an arch* DB *being given; to find the arch.* 6.

Draw Ct infinitely near CT, and Tn ⊥ Ct; and let CD $= r$, DT $= t$, T$t = \dot{t}$, DB $= z$, CT $= \sqrt{rr + tt}$.

The triangles CDT and Ttn are ſimilar, as alſo CBb and CTn; whence CT : r :: $\dot{t}$: nT $= \frac{r\dot{t}}{\mathrm{CT}}$, and

FIG. 6. and CT : nT or $\frac{r\dot{t}}{\text{CT}}$ :: r : Bb $= \frac{rr\dot{t}}{\text{CT}^2}$; that is, $\dot{z} = \frac{rr\dot{t}}{rr+tt}$.

And the fluent is $z = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} +$ &c; that is, arch DB $= t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8}$ &c.

Cor. 1. *In any arch a, the tangent is* $= a + \frac{a^3}{3rr} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} + \frac{1382a^{11}}{155925r^{10}} + \frac{21844a^{13}}{6081075r^{12}} + \frac{929569a^{15}}{638512875r^{14}} +$ &c.

This will appear by reverſion of the ſeries $a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4}$ &c. Or rather by multiplying the ſine by the radius, and dividing by the coſine: thus

$$\frac{ra - \frac{a^3}{6r} + \frac{a^5}{120r^3} \text{ \&c}}{r - \frac{a^2}{2r} + \frac{a^4}{24r^3} \text{ \&c}} = \text{tangent.}$$

Cor. 2. *The cotangent of an arch a is* $= \frac{rr}{a} - \frac{a}{3} - \frac{a^3}{45r^2} - \frac{2a^5}{945r^4} - \frac{a^7}{4725r^6} - \frac{2a^9}{93555r^8} - \frac{1382a^{11}}{638512875r^{10}} - \frac{4a^{13}}{18243225r^{12}} -$ &c, *as will appear by dividing radius ſquare by the value of the tangent in Cor.* 1. *Or the coſine by the ſine, and then multiplying by radius.*

Cor. 3. *Hence alſo if τ be the cotangent of an arch, then the arch* $= \frac{rr}{\tau} - \frac{r^4}{3\tau^3} + \frac{r^6}{5\tau^5} - \frac{r^8}{7\tau^7} + \frac{r^{10}}{9\tau^9} -$ &c. For $t = \frac{rr}{\tau}$.

PROP.

PROP. XVI.

FIG. 6.

The ſecant CT *of an arch* DB *being given; to find the arch.*

Let CD $= r$, CT $= ſ$, $nt = \dot{ſ}$, DT $= \sqrt{ſſ - rr}$, DB $= z$; then by ſimilar triangles, DT $: r :: \dot{ſ} :$ nT $= \frac{r\dot{ſ}}{\mathrm{DT}}$, and $ſ : r ::$ nT or $\frac{r\dot{ſ}}{\mathrm{DT}} : \dot{z} = \frac{rr\dot{ſ}}{ſ \times \mathrm{DT}}$, or $\dot{z} = \frac{rr\dot{ſ}}{ſ\sqrt{ſſ - rr}}$; and the Fluent is $z = \frac{rr}{ſ} - \frac{r^4}{2.3ſ^3} - \frac{3r^6}{2.4.5ſ^5} - \frac{3.5r^8}{2.4.6.7ſ^7}$ &c. but in D, $z = 0$, and $ſ = r$, therefore the Fluent corrected is, arch DB or $z = r \times : \frac{ſ - r}{ſ} + \frac{ſ^3 - r^3}{2.3ſ^3} + \frac{3.\overline{ſ^5 - r^5}}{2.4.5ſ^5} + \frac{3.5.\overline{ſ^7 - r^7}}{2.4.6.7ſ^7} +$ &c.

Or thus, In the point F, $z =$ a quadrant Q; and $ſ =$ infinity. Therefore the whole ſeries $= 0$. Therefore by correction $z - Q = 0 - \frac{rr}{ſ} - \frac{r^4}{2.3ſ^3}$ &c. whence arch z or DB $= Q - \frac{rr}{ſ} - \frac{r^4}{2.3.ſ^3} - \frac{3r^6}{2.4.5.ſ^5} - \frac{3.5r^8}{2.4.6.7ſ^7} -$ &c.

Cor. 1. *If* σ *be the coſecant of an arch, then the arch is* $= \frac{rr}{\sigma} + \frac{r^4}{2.3\sigma^3} + \frac{3r^6}{2.4.5\sigma^5} + \frac{3.5r^8}{2.4.6.7\sigma^7} +$ &c. *for this is the complement of the arch whoſe ſecant is* σ, *by this prop.*

Cor. 2. *In any arch* a, *the ſecant* $= r + \frac{aa}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} + \frac{50521a^{10}}{3628800r^9} + \frac{540553a^{12}}{95800320r^{11}} +$ &c.

For radius ſquare (rr), divided by the coſine $(r - \frac{aa}{2} + \frac{a^4}{24r^3})$ &c. is $=$ ſecant.

Cor. 3. *The coſecant of an arch a is* $= \frac{rr}{a} + \frac{a}{6} + \frac{7a^3}{360r^2} + \frac{31a^5}{15120r^4} + \frac{127a^7}{604800r^6} + \frac{73a^9}{3421440r^8} + \frac{1414477a^{11}}{653837184000r^{10}} +$ &c. *for radius ſquare divided by the ſine = coſecant.*

SCHOLIUM.

If an arch be given in degrees, its length may be ſound thus. When the radius of a circle is 1, half the circumference is 3.14159265358979, therefore $\frac{3.14159\ \&c.}{180} = ,01745329251994 =$ length of 1 degree. Therefore if r be the *radius* of any circle, then $r \times ,01745$ &c = length of the arch of 1 degree in that circle. Conſequently $r \times ,01745329$ &c $\times$ *number of degrees* and decimal parts, gives the length of the arch. And this muſt be taken for a or z in the foregoing propoſitions.

$$\text{Degrees} = \frac{arch}{.0174533r} = \frac{arch \times 57.29578}{r}.$$

SECT. III.

The calculation of logarithmic ſines, tangents, and ſecants.

Tho' the finding the logarithmic ſines, tangents, c, is no more than finding the logarithms of the ıtural ones by a table of logarithms: yet theſe garithmic ſines, tangents, &c, may be found withıt the table of logarithms, or elſe without the naıral ſines or tangents, and ſometimes without either, r having only the arch given.

The logarithmic or artificial ſines, tangents and cants are calculated to the radius 1 with 10 cyphers ınext, viz. 10000000000; ſo that the log. radius ill therefore be 10. But in calculating theſe ſines, ıgents, or ſecants, we can more eaſily compute em for the radius 1, and then adding 10, gives the g. ſine, tangent, or ſecant, to the radius of the bles. Therefore,

In a circle whoſe radius is 1, the log. of that raus is 0, and the length of an arch of 1 degree $\frac{3.1415926536}{180}$ =,01745329252; and this numr multiplied into any number of degrees, gives the length of the arch of theſe degrees, for the radius 1: and this length muſt be uſed in the following propoſitions, when the degrees are given.

Now if you put M =,4342944819, and having any quantity given (expreſſing a ſine, tangent, &c,) you have no more to do but to divide the fluxion of that quantity by the quantity itſelf, and find the fluent by infinite ſeries, which multiplied by M is the logarithm of the quantity required.

PROP. XVII.

To find the logarithmic or artificial ſine of the arch x.

Let $y =$ nat. ſine of x, then by Prop. XII, $y = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$ &c, where $r = 1$. And $\dot{y} = \dot{x} - \frac{x^2\dot{x}}{2} + \frac{x^4\dot{x}}{24} - \frac{x^6\dot{x}}{720}$ &c; then by diviſion $\frac{\dot{y}}{y} = \frac{\dot{x}}{x} - \frac{x\dot{x}}{3} - \frac{x^3\dot{x}}{45} - \frac{2x^5\dot{x}}{945} - \frac{x^7\dot{x}}{4725}$ &c; and Fluent of $\text{M}\frac{\dot{y}}{y}$, or the log. $y =$ log. $x - \text{M} \times : \frac{x^2}{6} + \frac{x^4}{180} + \frac{x^6}{2835} + \frac{x^8}{37800} +$ &c; to which add 10, and you have the logarithmic ſine in the tables; and that without knowing the natural ſine.

Otherwiſe.

Let $z = \frac{1-y}{1+y}$, where $y =$ nat. ſine; then will $y = \frac{1-z}{1+z}$, and $\frac{\dot{y}}{y} = \frac{-2\dot{z}}{1-zz} = -2 \times : \dot{z} + z^2\dot{z} + z^4\dot{z} + z^6\dot{z}$ &c; whence Fl. $\text{M}\frac{\dot{y}}{y}$, or log. $y = -2\text{M} \times : z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{7}z^7$ &c. And the log. ſine of the tables $= 10 - 2\text{M} \times : z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{7}z^7$ &c; which is had without the table of logarithms.

Or thus.

Let $y =$ nat. ſine, $z =$ coſine. Then $y = \sqrt{1-zz}$, and $\frac{\dot{y}}{y} = \frac{-z\dot{z}}{1-zz} = -z\dot{z} - z^3\dot{z} - z^5\dot{z}$ &c; whence Fl. $\frac{\text{M}\dot{y}}{y} = -\frac{1}{2}\text{M} \times : z^2 + \frac{z^4}{2} + \frac{z^6}{3} + \frac{z^8}{4}$ &c; and log. ſine of the tables $= 10 - \frac{1}{2}\text{M} \times : z^2 + \frac{1}{2}z^4 + \frac{1}{3}z^6 + \frac{1}{4}z^8$ &c.

And thus if you have any other line or quantity in the circle given, by which you can compute the ſine, you may, from that, find the log. ſine. And the like may be done for the tangent, or ſecant.

Cor. *Hence if* s = *log. ſine of* $\frac{1}{2}$ A. *half an arch given; then the log. verſed ſine of* A $= .3010300 + 2s - 10$, *for the tables.* For rad. × verſed ſine = 2 ſine ſquare of the half arch, by Cor. 2 Prop. II; and $.3010300 =$ log. 2.

Or thus. Log. verſ. of arch $x = 10.3010300 + 2 \log. \frac{1}{2}x - 2\mathrm{M} \times : \frac{x^2}{4.6} + \frac{x^4}{4^2.180} + \frac{x^6}{4^3.2835} + \frac{x^8}{4^4.37800}$ &c: for s (or the log. ſine of $\frac{1}{2}x$) $= 10 + \log. \frac{1}{2}x, - \mathrm{M} \times : \frac{x^2}{4.6} + \frac{x^4}{16.180}$ &c.

Or laſtly. If z = coſine, then log. verſ. $= 10 - \mathrm{M} \times : z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5}$ &c. For $1 - z$ = verſed ſine, and its fluxion $\frac{-\dot{z}}{1-z} = -\dot{z} - z\dot{z} - z^2\dot{z}$ &c; whoſe Fluent is $- \mathrm{M} \times : z + \frac{1}{2}z^2 + \frac{1}{3}z^3$ &c. to which add 10.

P R O P. XVIII.

The arch x being given, to find the log. coſine.

By Cor. I Pr. XII the coſine $y = 1 - \frac{xx}{2} + \frac{x^4}{24} - \frac{x^6}{720}$ &c; therefore $\frac{\dot{y}}{y} = - x\dot{x} - \frac{1}{3}x^3\dot{x} - \frac{2}{15}x^5\dot{x} - \frac{17}{315}x^7\dot{x}$ &c; and the Fl. $\frac{\mathrm{M}\dot{y}}{y}$ or log. $y = - \mathrm{M} \times : \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{17}{2520}x^8$ &c. And the tab. log. coſ. of the arch x is $= 10 - \mathrm{M} \times : \frac{1}{2}x^2 + \frac{1}{12}x^4$

$\frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{17}{2520}x^8 +$ &c. And that without either the nat. coſine or the table of logarithms.

Or the log. coſine may be found in the ſame manner as the ſine, in the two laſt methods in Pr. XVII.

PROP. XIX.

Given an arch z, and the ſine and coſine of an arch A; *to find the log. ſine and coſine of the ſum* $\overline{A+z}$, *and alſo of the difference* $\overline{A-z}$.

Let the nat. ſine and coſine of A be s, c; *and* $y =$ S. $\overline{A+z}$. Then by Prop. XIII, $y = s + \frac{cz}{1} - \frac{sz^2}{2} - \frac{cz^3}{6} + \frac{sz^4}{24}$ &c; and $\frac{\dot y}{y} = \frac{c\dot z}{s} - \frac{z\dot z}{ss} + \frac{cz^2\dot z}{s^3} - \frac{1+2cc}{3s^4}z^3\dot z$ &c; and the Fl. $M\frac{\dot y}{y}$ or log. $y = M \times : \frac{cz}{s} - \frac{z^2}{2ss} + \frac{cz^3}{3s^3} - \frac{1+2cc}{12s^4}z^4$ &c. But when $z = 0$, $y = s$, therefore by correction,

Log. y — log. $s = M \times : \frac{cz}{s} - \frac{z^2}{2ss}$ &c. Whence the

Log. S. $\overline{A+z} = log.\, s + M \times : \frac{cz}{s} - \frac{z^2}{2ss} + \frac{cz^3}{3s^3} - \frac{1+2cc}{12s^4}z^4$ &c. And after the ſame manner

Log. S. $\overline{A-z} = log.\, s - M \times : \frac{cz}{s} + \frac{z^2}{2ss} + \frac{cz^3}{3s^3} + \frac{1+2cc}{12s^4}z^4$ &c.

Log. coſ. $\overline{\text{A} + z} = log.\, c - \text{M} \times : \frac{sz}{c} + \frac{z^2}{2cc} + \frac{sz^3}{3c^3} + \frac{1 + 2ss}{12cc} z^4$ &c.

Log. coſ. $\overline{\text{A} - z} = log.\, c + \text{M} \times : \frac{sz}{c} - \frac{z^2}{2cc} + \frac{sz^3}{3c^3} - \frac{1 + 2ss}{12c^4} z^4$ &c.

PROP. XX.

The arch x being given, to find its log. tangent.

By Cor. 1 Pr. XV the tangent $t = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7$ &c; therefore $\frac{\dot{t}}{t} = \frac{\dot{x}}{x} + \frac{2}{3}x\dot{x} + \frac{14}{45}x^3\dot{x} + \frac{124}{945}x^5\dot{x}$ &c; whence log. tan. $t =$ log. $x + \text{M} \times : \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8$ &c: to which add 10.

Otherwiſe.

To find the log. tan. of the arch $\text{A} + x$, *the tan. of the arch* A *being given.*

Let tan. $\text{A} = a$, tan. $x = t$, $y =$ tan. $\overline{\text{A} + x}$. Then by Prop. VIII, $y = \frac{a + t}{1 - at}$, and $\dot{y} = \frac{\overline{1 + aa}\,\dot{t}}{\overline{1 - at}^2}$, and $\frac{\dot{y}}{y} = \frac{\overline{1 + aa}\,.\,\dot{t}}{a + \overline{1 - aa}\,.\,t - att}$. Let $b = \frac{1 + aa}{a}$, $c = \frac{1 - aa}{a}$, then $\frac{\dot{y}}{y} = b \times : \dot{t} - ct\dot{t} + \overline{1 + cc}\,.\,tt\dot{t} - \overline{2c + c^3}\,.\,t^3\dot{t}$ &c; and Fl. $\text{M}\frac{\dot{y}}{y}$ or log. $y = b\,\text{M} \times :$

$t - \frac{1}{2}ctt + \frac{1 + cc}{3}t^3 - \frac{2c + c^3}{4}t^4$ &c. But when x and $t = 0$, $y = a$; then by correction, log. y — log. $a = b\text{M} \times : t - \frac{1}{2}ctt$ &c; that is,

Log. tan. $\overline{\text{A} + x} =$ *log. tan.* $\text{A} + b\text{M} \times : t - \frac{1}{2}ctt + \frac{1 + cc}{3}t^3 - \frac{2c + c^3}{4}t^4$ &c. And the ſame for the *log. tan.* $\overline{\text{A} - x}$, changing the ſines of the odd powers of t.

But if $\text{A} = 45°$, then $a = 1$, and $\frac{\dot{y}}{y} = \frac{2\dot{t}}{1 - tt} = 2\dot{t} + 2t^2\dot{t} + 2t^4\dot{t}$ &c. And conſequently *log.* y or Fl. $\frac{\text{M}\dot{y}}{y} = 2\text{M} \times : t + \frac{1}{3}t^3 + \frac{1}{5}t^5$ &c. That is,

Log. tan. $\overline{45 + x} = 2\text{M} \times : t + \frac{t^3}{3} + \frac{t^5}{5} + \frac{t^7}{7}$, &c.

But if inſtead of t you put its value $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$ &c, and the fluxion for the fluxion, then will $\frac{2\dot{t}}{1 - tt} = 2 \times : \dot{x} + 2x^2\dot{x} + \frac{10}{3}x^4\dot{x} + \frac{244}{45}x^6\dot{x}$ &c; whence log. y, or *log. tan.* $\overline{45 + x} = 2\text{M} \times : x + \frac{2}{3}x^3 + \frac{2}{3}x^5 + \frac{244}{315}x^7$ &c. And likewiſe

Log. tan. $\overline{45 - x} = -2\text{M} \times : t + \frac{t^3}{3} + \frac{t^5}{5}$ &c. $= -2\text{M} \times : x + \frac{2}{3}x^3 + \frac{2}{3}x^5$ &c. To each of which add 10 for the tabular log. tangent.

Cor. *The log. cot. of* $x = 10 - \log. x - M \times : \frac{x^2}{3} + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8$ &c.

And log. cot. $\overline{A + x}$ is the ſame as the *log. tan.* of $\overline{A - x}$, found before.

PROP. XXI.

To find the logarithmic ſecant of a given arch.

Let x be the arch, y its ſecant; then by Cor. 2 Prop. XVI, $y = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6$ &c; whence $\frac{\dot{y}}{y} = x\dot{x} + \frac{1}{3}x^3\dot{x} + \frac{2}{15}x^5\dot{x} + \frac{17}{315}x^7\dot{x}$ &c; then log. y or Fl. $M\frac{\dot{y}}{y} = M \times : \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31}{14175}x^{10}$ &c; *hence the tab. log. ſecant* $= 10 + M \times : \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520}$ &c.

Or thus.

To find the log. ſecant of $\overline{45 + z}$, let log. ſecant $45 = s$. By Prop. XIX, log. coſ. $45 + z =$ log. coſ. $45 - M \times : \frac{sz}{c} + \frac{z^2}{2cc}$ &c $=$ (becauſe $s = c$, $r = 1$) log. coſ. $45 - M \times : z + zz$ &c; but by Cor. 2 Pr. I, ſecant $\times$ coſine $= 1$. Or log. ſecant $= -$ log. coſine. Therefore, log. ſec. $45 + z =$ log. ſec. $45 + M \times : z + zz +$ &c; that is,

Log. ſec. $\overline{45 + z} = s + M \times : z + z^2 + \frac{2}{3}z^3 + \frac{1}{3}z^4$ &c.

Cor. *The log. coſecant of an arch x is* $= 10 - \log. x + M \times : \frac{x^2}{6} + \frac{x^4}{180} + \frac{x^6}{2835} + \frac{x^8}{37800}$ &c. For log. coſecant $= -$ log. ſine, when radius is 1.

Or log. coſecant of $\overline{45+z}$, *or log. ſecant of* $\overline{45-z}$ $= s - \mathrm{M} \times : z - z^2 + \frac{2}{3}z^3 - \frac{1}{3}z^4$ &c, by this prop. changing the ſigns of the odd powers of z.

SCHOLIUM I.

Having the log. ſine, tangent or ſecant given, the arch belonging thereto may be found by the reverſion of ſeries. As ſuppoſe there is given T, the *log. tan.* $\overline{45+x}$, then $\mathrm{T} = 10 + 2\mathrm{M} \times : x + \frac{2}{3}x^3 + \frac{2}{5}x^5$ &c, by Pr. XX; and $x + \frac{2}{3}x^3 + \frac{2}{5}x^5$ &c $= \frac{\mathrm{T}-10}{2\mathrm{M}} = l$ ſuppoſe; then by reverſion $x = l - \frac{2}{3}l^3 + \frac{14}{15}l^5$ &c; and $\frac{x}{,0174532}$ = degrees in x.

Again, let there be given K, the log. ſecant of $\overline{45+z}$, then by Prop. XXI, $\mathrm{K} = s + \mathrm{M} \times : z + z^2 + \frac{2}{3}z^3$ &c; then $z + z^2 + \frac{2}{3}z^3$ &c $= \frac{\mathrm{K}-s}{\mathrm{M}} = l$. Then by reverſion, $z = l - ll + \frac{4}{3}l^3$ &c; which divided by ,0174532 will give the degrees in z.

SCHOLIUM II.

In any arch let log. radius $= r = 10$.

$s =$ log. ſine, $t =$ log. tangent, $\textit{ſ} =$ log. ſecant,
$c =$ log. coſine, $\tau =$ log. cotangent, $\sigma =$ log. coſecant.

Then

$s = c + t - r = t + r - \textit{ſ} = c + r - \tau = 2r - \sigma.$
$c = s + r - t = 2r - \textit{ſ} = s + \tau - r = \tau + r - \sigma.$
$t = s + r - c = 2r - \tau.$
$\tau = c + r - s = 2r - t.$
$\textit{ſ} = t + r - s = 3r - s - \tau = \sigma + r - \tau = 2r - c.$
$\sigma = \tau + r - c = 3r - c - t = \textit{ſ} + r - t = 2r - s.$

And log. verſ. = 2log. ſine of $\frac{1}{2}$arch — 9.6989700.

And log. verſ. ſup. = 2log. coſ. $\frac{1}{2}$arch — 9.6989700.

By what has been delivered in the two laſt ſections, the ſines, tangents, and ſecants, of all arches, whether natural or artificial, may eaſily be found; and from thence the table of ſines and tangents may be conſtructed, with great eaſe and expedition. Or any particular numbers in the tables may be computed anew, and any errors corrected therein. But the nat. ſines and tangents being firſt found, the artificial ones are moſt eaſily had by a table of logarithms; for this reaſon I have not continued the ſeveral ſeries in this laſt ſection to any great number of places; intending rather to give the reader the principles of calculation than the calculation itſelf; ſince in all probability there will be few perſons who will take the pains to calculate them anew. But if any has a mind to do it, he may himſelf continue any of theſe ſeries as far as he pleaſes.

SECT. IV.

The calculation of the ſines, coſines, cords, &c. of multiple arches.

PROP. XXII.

FIG. 19. *If a trapezoid* BCDE, *whoſe ſides* BE, CD *are parallel, be inſcribed in a circle; and the cords* BD, CE *be drawn; I ſay,* $BD^2 - DE^2 = BE \times CD$.

Make angle $EBO = CBD$, then $EBD = CBO$; and ſince $BEO = CDB$, therefore the triangles BCD and BOE are ſimilar, and $BD : CD :: BE : OE$, and $BD \times OE = BE \times CD$. Alſo ſince angle $BCO = BDE$, the triangles BCO and BDE are ſimilar, whence $BD : DE :: BC : CO$, and $BD \times CO = DE \times BC = DE^2$. Therefore $BD \times \overline{OE + OC} = BE \times CD + DE^2$; that is, $BD \times CE$ or $BD^2 = BE \times CD + DE^2$, or $BE \times CD = BD^2 - DE^2 = \overline{BD + DE} \times \overline{BD - DE}$.

Cor. 1. *If* A *be any arch, and* BC *or* DE $= n$ *times* A, *and* BD $= m$ *times* A. *Then,*

As cord of $\overline{m - n} \,.\, A$:
Cord $.\, mA$ + *cord* $.\, nA$: :
Cord $.\, mA$ — *cord* $.\, nA$:
Cord $.\, \overline{m + n} \,.\, A$.

For CD is the cord of $m - n$ times A, and BE of $\overline{m + n} \times A$.

Cor. 2. *As cord of* A :
Cord $.\, \overline{n + 1} \,.\, A$ + *cord* $.\, nA$: :
Cord $.\, \overline{n + 1} \,.\, A$ — *cord* $.\, nA$:
Cord $.\, \overline{2n + 1} \,.\, A$.

This appears by putting $m = n + 1$, or $CD = A$.

FIG. 19.

Cor. 3. *As cord of* $\overline{m-1}$. A :
Cord . *m*A + *cord* . A ::
Cord . *m*A — *cord* . A :
Cord $\overline{m+1}$. A.

This appears by putting $n = 1$, or CB or DE = A.

Cor. 4. *If three arches* CD, BCD, BCDE *are in arithmetic progreſſion,* $BD^2 - BC^2 = CD \times BE$.

PROP. XXIII.

In the circle AEB, *whoſe diameter is* AB, *if the arches* AD, EF, FG *be taken equal to one another, and the cords* BD, BE, BF, BG *drawn. I ſay,*

As radius CB : *cord* BD :: *ſo the middle cord* BF : BE + BG, *the ſum of the extreme cords.*

For produce BE to H, and make FH = FB; then ∠BHF = HBF = FBG; and draw the cords EF, FG. Then the angle BEF = BFG + FBG, both ſtanding on the ſame arch FGB. Alſo BEF = EFH + EHF = EFH + FBG, therefore EFH = BFG. And the triangles EFH, FBG are ſimilar and equal. Whence HE = GB; and HB = EB + BG. Alſo the triangles DCB and HFB are ſimilar, therefore BC : BD :: BF : BH or EB + BG.

Cor. *If arch* AD = A, *arch* BF = $n \times$ A. *Then*
As rad :
Cord . *ſup.* A ::
Cord . *n*A :
Cord . $\overline{n-1}$. A + *cord* . $\overline{n+1}$. A.

And the ſame holds in reſpect of the cords of the ſupplements of theſe arches, if AF be ſuppoſed to be made equal to *n*A.

SCHOLIUM.

If the cord BG lie on the other ſide of B, then BG is negative.

PROP. XXIV.

The ſine and coſine of an arch being given; to find the ſine of n times that arch.

Let $x =$ coſine, $y =$ ſine, given. And A the given arch, and at preſent let radius $= 1$. Then by Cor. 7 Prop. III, $2x \times \mathrm{S}.n\mathrm{A} = \mathrm{S}.\overline{n-1}.\mathrm{A} + \mathrm{S}.\overline{n+1}.\mathrm{A}$; whence $\mathrm{S}.\overline{n+1}.\mathrm{A} = 2x \times \mathrm{S}.n\mathrm{A} - \mathrm{S}.\overline{n-1}.\mathrm{A}$. Or, becauſe $xx + yy = 1$, $\mathrm{S}.\overline{n+1}.\mathrm{A} = 2x \times \mathrm{S}.n\mathrm{A} - \overline{xx+yy} \times \mathrm{S}.\overline{n-1}.\mathrm{A}$. That is, if any ſine be multiplied by $2x$, and the next preceding one multiplied by $xx + yy$, and ſubtracted from it, gives the next following line. Thus you will have

$$\begin{aligned} &\mathrm{S}.\ \mathrm{A} = y. \\ &\mathrm{S}.\ 2\mathrm{A} = 2xy. \\ &\mathrm{S}.\ 3\mathrm{A} = 3x^2y - y^3. \\ &\mathrm{S}.\ 4\mathrm{A} = 4x^3y - 4xy^3. \\ &\mathrm{S}.\ 5\mathrm{A} = 5x^4y - 10x^2y^3 + y^5. \\ &\text{\&c.} \end{aligned}$$

In what follows, if a denotes any variable quantity, then I denote its firſt and ſecond ſucceeding values by a_1, a_2, &c; and its firſt and ſecond preceding values by a_{-1}, a_{-2}, &c; and ſo of others. And with theſe I proceed according to the method of increments in calculating the following propoſitions.

Suppoſe then any arch, as $n\mathrm{A}$, is repreſented in general thus, $\mathrm{S}.n\mathrm{A} = nx^{n-1}y - ax^{n-3}y^3 + bx^{n-5}y^5 -$ &c: then according to the conſtruction of ſines, before mentioned, we ſhall have

$2nx^ny$

$$\left\{\begin{array}{l} 2nx^ny - 2ax^{n-2}y^3 + 2bx^{n-4}y^5 - \&c = \text{S}.n\text{A}\times 2x, \\ \left.\begin{array}{l} -\underset{-1}{n}x^ny + \underset{-1}{a}x^{n-2}y^3 - \underset{-1}{b}x^{n-4}y^5 - \&c \\ \qquad\qquad -\underset{-1}{n} \qquad\qquad\quad +\underset{-1}{a} \end{array}\right\} = \overline{xx+yy}\times \text{S}.\overline{n-1}.\text{A}. \end{array}\right.$$

$$= \overline{n+1}.x^ny - \underset{1}{a}x^{n-2}y^3 + \underset{1}{b}x^{n-4}y^5 - \&c = \text{S}.\overline{n+1}.\text{A}.$$

Hence equating the coefficients of the homologous powers, $n+1 = 2n - \underset{-1}{n}$, and $1 = n - \underset{-1}{n}$; that is, $n = 1$. Likewiſe $\underset{1}{a} = 2a - \underset{-1}{a} + \underset{-1}{n}$, or $\underset{2}{a} = 2\underset{1}{a} - a + n$, whence $\underset{2}{a} - 2\underset{1}{a} + a = n$; that is, $\underset{\cdot\cdot}{a} = n$. And the integral is $\underset{\cdot}{a} = \frac{\underset{-1}{n}\,n}{2}$, and the integral of this is $a = \frac{\underset{-2}{n}\,\underset{-1}{n}\,n}{2.3}$ (for when $n = 3$, $a = \frac{\underset{-2}{n}\,\underset{-1}{n}\,n}{2.3} = 1$). Again, $\underset{1}{b} = 2b - \underset{-1}{b} + \underset{-1}{a}$, or $\underset{2}{b} = 2\underset{1}{b} - b + a$, whence $\underset{2}{b} - 2\underset{1}{b} + b = a$, or $\underset{\cdot\cdot}{b} = a = \frac{\underset{-2}{n}\,\underset{-1}{n}\,n}{2.3}$, whence $\underset{\cdot}{b} = \frac{\underset{-3}{n}\,\underset{-2}{n}\,\underset{-1}{n}\,n}{2.3.4}$, and $b = \frac{\underset{-4}{n}\,\underset{-3}{n}\,\underset{-2}{n}\,\underset{-1}{n}\,n}{2.3.4.5}$. In like manner $\underset{\cdot\cdot}{c} = b$, and $c = \frac{\underset{-6}{n}\,\underset{-5}{n}\,\underset{-4}{n}\,\underset{-3}{n}\,\underset{-2}{n}\,\underset{-1}{n}\,n}{2.3.4.5.6.7}$, &c. *Whence* S.$n$A $= nx^{n-1}y - \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}x^{n-3}y^3 + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.2.3.4.5}x^{n-5}y^5$ &c. Or putting r = rad. S.nA $= \frac{nx^{n-1}y}{r^{n-1}} - \frac{\overline{n-1}.\overline{n-2}.yy}{2.3xx}\text{A} - \frac{\overline{n-3}.\overline{n-4}.y^2}{4.5xx}\text{B} - \frac{\overline{n-5}.\overline{n-6}.yy}{6.7xx}\text{C}$ &c: where A, B, C, &c, are the preceding terms with their ſigns.

Cor. *If you ſuppoſe r to be the diameter, y the cord of an arch, x the cord of the ſupplement; then the foregoing ſeries will be the cord of the multiple arch.*

PROP. XXV.

The ſine of an arch A *being given; to find the ſine of n* A, *or n times that arch.*

Let $s =$ ſine of A, $z =$ ſine of nA. Proceeding as in Prop. XI, we ſhall have $\dot{A} = \frac{r\dot{s}}{\sqrt{rr - ss}}$, and $n\dot{A} = \frac{r\dot{z}}{\sqrt{rr - zz}} = \frac{rn\dot{s}}{\sqrt{rr - ss}}$; whence $\overline{rr - ss} \times \dot{z}^2 = nn\dot{s}^2 \times \overline{rr - zz}$, this in fluxions (putting $\dot{s}$ conſtant), will be $\overline{rr - ss} \times 2\dot{z}\ddot{z} - 2s\dot{s}\dot{z}^2 = -2nn\dot{s}^2 z\dot{z}$, and $rr\ddot{z} - ss\ddot{z} - s\dot{s}\dot{z} + nnz\dot{s}^2 = 0$.

Aſſume $z = as + bs^3 + cs^5 + ds^7$ &c.
then $\dot{z} = a\dot{s} + 3bss\dot{s} + 5cs^4\dot{s} + 7ds^6\dot{s}$ &c.
and $\ddot{z} = 2.3bs\dot{s}^2 + 4.5cs^3\dot{s}^2 + 6.7ds^5\dot{s}^2$ &c.

Then by ſubſtitution,

$$\left.\begin{array}{r|llll} rr\ddot{z} & 2.3brrs\dot{s}^2 & + 4.5crrs^3\dot{s}^2 & + 6.7drrs^5\dot{s}^2 & \text{\&c.} \\ -ss\ddot{z} & & -2.3bs^3\dot{s}^2 & -4.5cs^5\dot{s}^2 & \text{\&c.} \\ -s\dot{s}\dot{z} & -as\dot{s}^2 & -3bs^3\dot{s}^2 & -5cs^5\dot{s}^2 & \text{\&c.} \\ +nnz\dot{s}^2 & +nnas\dot{s}^2 & +nnbs^3\dot{s}^2 & +nncs^5\dot{s}^2 & \text{\&c.} \end{array}\right\} = 0.$$

Now when A is infinitely ſmall, $s =$ A, and $z = n$A $= ns = as$, therefore $a = n$. And equating the coefficients, $b = -\frac{nn - 1}{2.3rr}a$, $c = -\frac{nnb - 2.3b - 3b}{4.5rr} = -\frac{nn - 3.3}{4.5rr}b$, $d = -\frac{nn - 5.5}{6.7rr}c$, $e = -\frac{nn - 7.7}{8.9rr}d$, &c: therefore $z = ns - \frac{nn - 1}{2.3rr}as^3 - \frac{nn - 3.3}{4.5rr}bss$

$\frac{nn-3.3}{4.5rr}bs^5$ &c $= ns - \frac{n.\overline{nn-1}}{2.3rr}s^3 +$ $\frac{n.\overline{nn-1}.\overline{nn-9}}{2.3.4.5r^4}s^5$ &c; and ſince coſ. A $=$ $\sqrt{rr-ss} = r - \frac{ss}{2r} - \frac{s^4}{8r^3} - \frac{s^6}{16r^5}$ &c, therefore

$$\frac{z}{\sqrt{rr-ss}} = \frac{ns - \frac{n.\overline{nn-1}}{2.3rr}s^3 + \frac{n.\overline{nn-1}.\overline{nn-9}}{2.3.4.5r^4}s^5 \text{ \&c}}{r - \frac{ss}{2r} - \frac{s^4}{8r^3} \text{ \&c}}$$

$$= \frac{ns}{r} - \frac{nn-2.2}{2.3r^3}ns^3 + \frac{n.\overline{nn-2.2}.\overline{nn-4.4}}{2.3.4.5r^5}s^5 \text{ \&c;}$$

whence we have

z *or* $S.nA = ns - \frac{nn-1}{2.3rr}ssA - \frac{nn-2.3}{4.5rr}ssB - \frac{nn-5.5}{6.7rr}ssC - \frac{nn-7.7}{8.9rr}ssD$ &c: *where* A, B, C &c, *are the preceding terms with their ſigns; therefore if n be an odd number, the ſeries will be finite.*

Or $S.nA = \sqrt{rr-ss} \times : \frac{ns}{r} - \frac{nn-2.2}{2.3rr}ssA - \frac{nn-4.4}{4.5rr}ssB - \frac{nn-6.6}{6.7rr}ssC - \frac{nn-8.8}{8.9rr}ssD$ &c: *and therefore if n be an even number, the ſeries will be finite.*

Cor. *If s be the cord of an arch, x the cord of the ſupplement, d = diameter; then*

Cord of $nA = ns - \frac{nn-1}{2.3dd}ssA - \frac{n^2-3^2}{4.5.d^2}ssB - \frac{n^2-5^2}{6.7d^2}ssC - \frac{n^2-7^2}{8.9d^2}ssD$ &c: *or*

Cord of n arch $= \frac{nsx}{d} - \frac{n^2 - 2^2}{2.3d^2}s^2\text{A} - \frac{n^2 - 4^2}{4.5.d^2}s^2\text{B}$
$- \frac{n^2 - 6^2}{6.7d^2}s^2\text{C} -$ &c.

Scholium.

If z the ſine of an arch was given, and the ſine s, of the n^{th} part of the arch, was required; you'll have by reverſion of ſeries,

$$s = \frac{z}{n} + \frac{\overline{n^2 - 1} . z^2}{2.3r^2n^2}\text{A} + \frac{\overline{3n^2}| - 1}{4.5r^2n^2}\text{B}z^2 +$$
$$\frac{\overline{5n}|^2 - 1}{6.7r^2n^2}\text{C}z^2 \text{ \&c.}$$

PROP. XXVI.

The ſine and coſine of an arch being given; to find the coſine of n times that arch.

Let A $=$ arch, $y =$ ſine, $x =$ coſine. Radius $= 1$. Then by Cor. 7 Pr. III, $2x \times$ coſ. nA $=$ coſ. $\overline{n - 1}$.A $+$ coſ. $\overline{n + 1}$.A; therefore ſince $\overline{xx + yy} = 1$, coſ. $\overline{n + 1}$.A $= 2x \times$ coſ. nA $- \overline{xx + yy} \times$ coſ. $\overline{n - 1}$.A. Therefore if you multiply any coſine by $2x$, and from it ſubtract the foregoing one multiplied into $xx + yy$, the remainder is the next ſucceeding coſine. Hence.

Coſ. 0A $= r$
Coſ. 1A $= x$
Coſ. 2A $= xx - yy$
Coſ. 3A $= x^3 - 3xyy$
Coſ. 4A $= x^4 - 6x^2y^2 + y^4$
Coſ. 5A $= x^5 - 10x^3y^2 + 5xy^4$
Coſ. 6A $= x^6 - 15x^4y^2 + 15x^2y^4 - y^6$
Coſ. 7A $= x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6$,
&c. Therefore in general, let the

Coſ. $\overline{n-1}.A = x^{n-1} - ax^{n-3}y^2 + bx^{n-5}y^4 - cx^{n-7}y^6$ &c.
Coſ. $nA = x^n - \underset{1}{a}x^{n-2}y^2 + \underset{1}{b}x^{n-4}y^4 - \underset{1}{c}x^{n-6}y^6$ &c.

Now if the latter be multiplied by $2x$, and the former by $xx + yy$, and this product ſubtracted from the firſt, you will have

$$\left.\begin{array}{rcccccccc} 2x^{n+1} & - & 2\underset{1}{a}x^{n-1}y^2 & + & 2\underset{1}{b}x^{n-3}y^4 & - & 2\underset{1}{c}x^{n-5}y^6 & \&c \\ -x^{n+1} & + & ax^{n-1}y^2 & - & bx^{n-3}y^4 & + & cx^{n-5}y^6 & \\ & - & x^{n-1}y^2 & + & ax^{n-3}y^4 & - & bx^{n-5}y^6 & \end{array}\right\}$$

$$= x^{n+1} - \underset{2}{a}x^{n-1}y^2 + \underset{2}{b}x^{n-3}y^4 - \underset{2}{c}x^{n-5}y^6 \ \&c$$

$=$ coſ. $\overline{n+1}.A$. Then comparing the coefficients, we have $\underset{2}{a} = 2\underset{1}{a} - a + 1$, that is, by the method of increments, $a + 2\underset{\cdot}{a} + \underset{\cdot\cdot}{a} = 2a + 2\underset{\cdot}{a} - a + 1$, or $\underset{\cdot\cdot}{a} = 1 = \underset{\cdot}{n}$; therefore $\underset{\cdot}{a} = n$, the integral, and $\underset{1}{a} = \frac{n\,\overline{n-1}}{2}$, for when $n = 2$, $\underset{1}{a}$ or $\frac{n\,\overline{n-1}}{2} = 1$. Likewiſe $\underset{2}{b} = 2\underset{1}{b} - b + a$, and $\underset{\cdot\cdot}{b} = a = \frac{\overline{n-2}\,\overline{n-1}}{2}$, and $\underset{\cdot}{b} = \frac{\overline{n-3}\,\overline{n-2}\,\overline{n-1}}{2\cdot3}$, and $b = \frac{\overline{n-4}\,\overline{n-3}\,\overline{n-2}\,\overline{n-1}}{2\cdot3\cdot4}$, (for when $n = 5$, $b = 1$,) and $\underset{1}{b} = \frac{n\,\overline{n-3}\,\overline{n-2}\,\overline{n-1}}{2\cdot3\cdot4}$. Again, $\underset{\cdot\cdot}{c} = \underset{\cdot}{b} = \frac{\overline{n-4}\cdots\overline{n-1}}{2\cdot3\cdot4}$, $\underset{\cdot}{c} = \frac{\overline{n-5}\cdots\overline{n-1}}{2\cdot3\cdot4\cdot5}$, and $c = \frac{\overline{n-6}\cdots\overline{n-1}}{2\ldots\ldots6}$, and therefore $\underset{1}{c} = \frac{n\ldots\overline{n-5}}{2\ldots\ldots6}$. Likewiſe $\underset{1}{d} = \frac{n\ldots\overline{n-7}}{2\ldots7}$, &c.

Then ſince $n = 1$, therefore $n, \underset{-1}{n}, \underset{-2}{n}$, &c, are equal to $n, n-1, n-2$, &c; whence putting r for radius,

$$\text{Coſ.}\; n\text{A} = \frac{1}{r^{n-1}} \times : x^n - \frac{n\cdot\overline{n-1}}{2} x^{n-2}y^2 +$$
$$\frac{n\cdot\overline{n-1}\cdot\overline{n-2}\cdot\overline{n-3}}{2\cdot3\cdot4} x^{n-4}y^4 -$$
$$\frac{n\cdot\overline{n-1}\cdot\overline{n-2}\cdot\overline{n-3}\cdot\overline{n-4}\cdot\overline{n-5}}{2\cdot3\cdot4\cdot5\cdot6.} x^{n-6}y^6$$

&c. Or

$$\text{Coſ.}\; n\text{A} = \frac{x^n}{r^{n-1}} - \frac{n\cdot\overline{n-1}}{1\cdot2}\text{A}\frac{yy}{xx} - \frac{\overline{n-2}\cdot\overline{n-3}}{3\cdot4}$$
$$\text{B}\frac{yy}{xx} - \frac{\overline{n-4}\cdot\overline{n-5}}{5\cdot6}\text{C}\frac{yy}{xx} - \frac{\overline{n-6}\cdot\overline{n-7}}{7\cdot8}\text{D}\frac{yy}{xx},$$

&c: *where* A, B, C &c, *are the foregoing terms with their ſigns.*

Cor. *If you ſuppoſe r the diameter, y the cord of an arch, x the cord of the ſupplement; then the foregoing ſeries will be the cord of the ſupplement of the multiple arch.*

PROP. XXVII.

The ſine s of an arch a being given; to find the coſine of na, or n times that arch.

Let $z =$ coſ. na, rad. $= 1$, then by Cor. 1 Pr. 12, $z = 1 - \frac{n^2a^2}{2} + \frac{n^4a^4}{24} - \frac{n^6a^6}{720} + \frac{n^8a^8}{40320}$ &c: but by Prop. 11, $a = s + \frac{s^3}{6} + \frac{3s^5}{40} + \frac{5s^7}{112}$ &c:

whence $aa = ss + \frac{s^4}{3} + \frac{8s^6}{45} + \frac{4s^8}{35}$ &c,

and

$$\begin{aligned} \text{and} \qquad a^4 &= s^4 + \tfrac{2}{3}s^6 + \tfrac{7}{15}s^8 \text{ \&c,} \\ a^6 &= \phantom{s^4 + \tfrac{2}{3}} s^6 + \phantom{\tfrac{7}{15}} s^8 \\ a^8 &= \phantom{s^4 + \tfrac{2}{3}s^6 + \tfrac{7}{15}} s^8 \\ &\text{\&c :} \end{aligned}$$

then all theſe values of the powers of *a* being ſubſtituted in the value of *z*, gives

$$\begin{aligned} z = 1 - \frac{nnss}{2} - \frac{n^2s^4}{6} - \frac{4n^2s^6}{45} - \frac{2n^2s^8}{35} &\text{ \&c,} \\ + \frac{n^4s^4}{24} + \frac{n^4s^6}{36} + \frac{7n^4s^8}{360} & \\ - \frac{n^6s^6}{720} - \frac{n^6s^8}{720} & \\ + \frac{n^8s^8}{40320} &\text{ \&c :} \end{aligned}$$

whence

$$z = 1 - \frac{n^2}{2}s^2 + \frac{nn.\overline{n^2 - 2^1}}{2 \cdot 3 \cdot 4}s^4 - \frac{n^2.\overline{n^2 - 2^2}.\overline{n^2 - 4^2}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}s^6.$$

&c ; that is, if r = radius,

$$Coſ.\ na = r - \frac{n^2s^2}{1.2r^2}\text{A} - \frac{n^2 - 2^2}{3 \cdot 4r^2}ss\text{B} - \frac{n^2 - 4^2}{5 \cdot 6r^2}ss\text{C} - \frac{n^2 - 6^2}{7 \cdot 8r^2}ss\text{D}$$

&c : *where* A, B, C, &c, *are the foregoing terms with their ſigns. Therefore, if n be an even number, the ſeries will be finite.*

And ſince $\sqrt{rr - ss} = r - \frac{ss}{2r} - \frac{s^4}{8r^3} - \frac{s^6}{16r^5} - \frac{5s^8}{128r^7}$ &c = coſ. A, therefore dividing the foregoing ſeries by this, and then multiply by $\sqrt{rr - ss}$, and you will have $coſ.\ na = \sqrt{rr - ss} \times :$

$$1 - \frac{n^2 - 1}{1.2r^3}ss\text{A} - \frac{n^2 - 3^2}{3.4r^2}ss\text{B} - \frac{n^2 - 5^2}{5.6r^2}ss\text{C} - \frac{n^2 - 7^2}{7 \cdot 8r^2}ss\text{D}$$

$\frac{n^2-7^2}{7.8r^2}$ ssd &c. *And therefore if n be an odd number, the ſeries will be finite.*

Cor. *The foregoing ſeries will expreſs the cord of the ſupplement of the n^{th} arch; putting s for the cord of the ſingle arch, and r for the diameter.*

PROP. XXVIII.

The cord of an arch, and the cord of its ſupplement being given; to find the cord of n times that arch.

Let A be the arch, y the cord, x the cord of the ſupplement, radius $= 1$; then by Cor. Pr. XXIII, $x \times$ cord . nA $=$ cord . $\overline{n-1}$. A $+$ cord . $\overline{n+1}$. A, and cord . $\overline{n+1}$. A $= x \times$ cord . nA $-$ cord . $\overline{n-1}$. A; that is, if any cord be multiplied by x, and the foregoing one ſubtracted, will give the next following cord. Whence

$$\begin{aligned}
&\textit{Cord}\ .\ \ \text{A} = y\\
&\textit{Cord}\ .\ 2\text{A} = xy\\
&\textit{Cord}\ .\ 3\text{A} = x^2y - y\\
&\textit{Cord}\ .\ 4\text{A} = x^3y - 2xy\\
&\textit{Cord}\ .\ 5\text{A} = x^4y - 3x^2y + y, \ \&c.
\end{aligned}$$

And in general, let the cord of nA be $= yx^{n-1} - ax^{n-3}y + bx^{n-5}y - cx^{n-7}y$ &c; then will

$$\left.\begin{aligned}
yx^n - ayx^{n-2} + byx^{n-4} - cyx^{n-6}\ \&\text{c}\\
- yx^{n-2} + \underset{-1}{a}yx^{n-4} - \underset{-1}{b}yx^{n-6}\ \&\text{c}
\end{aligned}\right\} =$$

$$yx^n - \underset{1}{a}yx^{n-2} + \underset{1}{b}yx^{n-4} - \underset{1}{c}yx^{n-6}\ \&\text{c} = \text{cord}\ .$$

$\overline{n+1}$. A. Then equating the coefficients $a = \underset{1}{a} + 1$, and $\underset{1}{a} - a$ or $\underset{\cdot}{a} = 1 = \underset{\cdot}{n}$, and the integral is $a = \underset{-2}{n}$ (for when $n = 3$, a or $\underset{-2}{n} = 1$). Again, $\underset{1}{b}$ or

$b +$

$b + \underset{\cdot}{b} = b + \underset{-1}{a}$, or $\underset{\cdot}{b} = \underset{-1}{a} = \underset{-3}{n}$, and $b = \frac{\overset{n}{-3}\overset{n}{-4}}{2}$; alſo $\underset{1}{c} = c + \underset{-1}{b}$, or $\underset{\cdot}{c} = \underset{-1}{b} = \frac{\overset{n}{-4}\overset{n}{-5}}{2}$, and $c = \frac{\overset{n}{-4}\overset{n}{-5}\overset{n}{-6}}{2 \cdot 3}$. In like manner $d = \frac{\overset{n}{-5}\overset{n}{-6}\overset{n}{-7}\overset{n}{-8}}{2 \cdot 3 \cdot 4}$, &c. Whence the *cord* . $n\text{A} = yx^{n-1} - \frac{n-2}{1}yx^{n-3} + \frac{\overline{n-3} \cdot \overline{n-4}}{1 \cdot 2}yx^{n-5} - \frac{\overline{n-4} \cdot \overline{n-5} \cdot \overline{n-6}}{1 \cdot 2 \cdot 3}yx^{n-7} + \frac{\overline{n-5} \cdot \overline{n-6} \cdot \overline{n-7} \cdot \overline{n-8}}{1 \cdot 2 \cdot 3 \cdot 4}yx^{n-9}$ &c; that is, putting radius $= r$, then

$$\textit{Cord} \;.\; n \textit{ arch} = \frac{yx^{n-1}}{r^{n-1}} - \frac{n-2}{1}\text{A}\frac{rr}{xx} - \frac{\overline{n-3} \cdot \overline{n-4}}{2 \cdot \overline{n-2}}\text{B}\frac{rr}{xx} - \frac{\overline{n-5} \cdot \overline{n-6}}{3 \cdot \overline{n-3}}\text{C}\frac{rr}{xx} - \frac{\overline{n-7} \cdot \overline{n-8}}{4 \cdot \overline{n-4}}\text{D}\frac{rr}{xx} - \frac{\overline{n-9} \cdot \overline{n-10}}{5 \cdot \overline{n-5}}\text{E}\frac{rr}{xx} \text{ \&c.}$$

Cor. *If* $r =$ *radius*, $y =$ *ſine*, $x =$ *coſine of an arch, then the ſine of the* n^{h} $\frac{2x^{n-1}}{r^{n-1}}y - \frac{n-2 \cdot rr}{4xx}\text{A} - \frac{\overline{n-3} \cdot \overline{n-4} \cdot rr}{4 \cdot 2 \cdot \overline{n-2} \cdot xx}\text{B} - \frac{\overline{n-5} \cdot \overline{n-6} \cdot rr}{4 \cdot 3 \cdot \overline{n-3} \cdot xx}\text{C} - \frac{n-7 \cdot n-8 \cdot rr}{4 \cdot 4 \cdot \overline{n-4} \cdot xx}\text{D}$ &c: this is plain by putting $\frac{1}{2}r$ for r in the foregoing ſeries.

PROP. XXIX.

The ſquare of the cord of an arch being given; to find the ſquare of the cord of n times the arch.

Let a, b, c be the cords of three arches in arithmetic progreſſion, rad. $= 1$, $x =$ cord of the common difference, and $\sqrt{4 - xx} =$ cord of the ſup. common difference. Then by Prop. XXIII, rad. × ſum of the cords of the extreme arches = cord mean × cord ſup. common difference; that is, $b\sqrt{4 - xx} = a + c$. And by Cor. 4 Pr. XXII, $ac = bb - xx$. Whence $4bb - bbxx = aa + cc + 2ac = aa + cc + 2bb - 2xx$, therefore $bb \times \overline{2 - xx} + 2xx - aa = cc$; that is, in any ſeries of the ſquares of cords, if the ſquare of any cord be multiplied by $-xx + 2$, and the preceding one ſubtracted, and $2xx$ added, you will have the ſquare of the next following cord. Thus, ſquare of the cord

of A $= xx$
2A $= -x^4 + 4x^2$
3A $= x^6 - 6x^4 + 9x^2$
4A $= -x^8 + 8x^6 - 20x^4 + 16x^2$
5A $= x^{10} - 10x^8 + 35x^6 - 50x^4 + 25x^2$
&c.

Therefore let any one in general be expreſſed thus,

$$x^v - ax^{v-2} + bx^{v-4} - cx^{v-6} \text{ \&c};$$

then the next ſucceeding one (changing the ſigns) will be $-\overset{v}{x^{1}} + a\overset{v-2}{x^{1}} - \underset{1}{b}\overset{v-4}{x^{1}} + \underset{1}{c}\overset{v-6}{x^{1}}$ &c $=$ (by conſtruction to)

$$\left.\begin{array}{rrrrr} -x^{v+2} & +\,ax^{v} & -\,bx^{v-2} & +\,cx^{v-4} \\ & +\,2x^{v} & -\,2ax^{v-2} & +\,2bx^{v-4} \\ & & +\,x^{v-2} & -\,\underset{-1}{a}x^{v-4} \end{array}\right\} \begin{array}{l}\text{\&c} + \\ 2xx.\end{array}$$

Then comparing the homologous terms, we have, $\nu_1 = \nu + 2$, or $\nu_1 - \nu$, that is, $\nu = 2 = 2n$, and $\nu = 2n$. Again, $a_1 = a + 2$, or $a_1 - a$ or $a = 2 = \nu$; whence $a = \nu = 2n$. Alſo $b_1 = b + 2a - 1$, or $b = 2a - 1 = 2\nu - \frac{1}{2}\nu = \nu\nu - \frac{1}{2}\nu$; therefore $b = \frac{\nu\nu_{-1}}{2} - \frac{\nu}{2}$.

Again, $c_1 = c + 2b - a_{-1}$, or $c = \nu\nu_{-1} - \nu - \nu_{-1} = \nu\nu_{-1} - 2\nu + 2 = \nu\nu_{-1} - 2\nu + \nu$, and $c = \frac{\nu\nu_{-2}\nu_{-1}}{2\cdot 3} - \frac{\nu\nu_{-1}}{2} + \nu$.

In like manner $d = 2c - b_{-1} = \frac{\nu\nu_{-2}\nu_{-1}}{3} - \frac{3\nu\nu_{-1}}{2} + \frac{9\nu}{2} - \frac{5\nu}{2}$; and $d = \frac{\nu\nu_{-3}\nu_{-2}\nu_{-1}}{2\cdot 3\cdot 4} - \frac{\nu\nu_{-2}\nu_{-1}}{2\cdot 2} + \frac{9\nu\nu_{-1}}{2.2.2} - \frac{5\nu}{2}$.

But ſince $\nu = 2$, $\nu_{-1} = \nu - 2$, $\nu_{-2} = \overline{\nu - 4}$, &c, we ſhall find $b = \nu \times \frac{\nu - 3}{2}$, $c = \nu \times \frac{\nu - 4}{2} \times \frac{\nu - 5}{3}$, $d = \nu \times \frac{\overline{\nu - 5} \times \overline{\nu - 6} \times \overline{\nu - 7}}{2\cdot 3\cdot 4}$ &c; *therefore if x is the cord of an arch, and b the cord of n times the arch, then will*

$$x^{2n} - 2nx^{2n-2} + 2n \times \frac{2n-3}{2} x^{2n-4} - 2n \times \frac{\overline{2n-4}}{2}$$

$$\frac{\overline{2n-4}\cdot\overline{2n-5}}{2\cdot 3}x^{2n-6}+2n\times\frac{\overline{2n-5}\cdot\overline{2n-6}\cdot\overline{2n-7}}{2\cdot 3\cdot 4}x^{2n-8}\ \&c=\pm bb,$$

according as n is odd or even; and the ſeries continued to n terms. The laſt term being always $nnxx$.

Or
$$\frac{x^{2n}}{r^{2n-2}}-\frac{2nr^2}{xx}\mathrm{A}-\frac{\overline{2n-3}}{2xx}rr\mathrm{B}-\frac{\overline{2n-4}\cdot\overline{2n-5}}{3\cdot\overline{2n-3}\cdot xx}rr\mathrm{C}-\frac{\overline{2n-6}\cdot\overline{2n-7}}{4\cdot\overline{2n-4}\cdot xx}rr\mathrm{D}-\frac{\overline{2n-8}\cdot\overline{2n-9}}{5\cdot\overline{2n-5}\cdot xx}rr\mathrm{E}-\frac{\overline{2n-10}\cdot\overline{2n-11}}{6\cdot\overline{2n-6}\cdot xx}rr\mathrm{F}-\ \&c=\pm bb.$$

Otherwiſe.

Since the law of the continuation of the ſeries of the ſquares of the cords is this, that any cord multiplied by $2-xx$, and the foregoing one ſubtracted, and $2xx$ added, gives the next. Therefore

$$\begin{array}{rl}
\square\ 1\ \text{Cord} & = xx\\
2. & = 4x^2-x^4\\
3. & = 9x^2-6x^4+x^6\\
4. & = 16x^2-20x^4+8x^6-x^8\\
5. & = 25x^2-50x^4+35x^6-10x^8+x^{10}\\
\&c. &
\end{array}$$

And in general $ax^2-bx^4+cx^6-dx^8$ &c $=$ ſquare of the cord of the n^{th} arch. Whence by the conſtruction,

$$\left.\begin{array}{l}
2ax^2-2bx^4+2cx^6-2dx^8\\
+\overset{1}{2}x^2-\overset{1}{a}x^4+\overset{1}{b}x^6-\overset{1}{c}x^8\\
-ax^2+\overset{1}{b}x^4-\overset{1}{c}x^6+\overset{1}{d}x^8
\end{array}\right\}=$$

$$\underset{2}{a}x^2-\underset{2}{b}x^4+\underset{2}{c}x^6-\underset{2}{d}x^8\ \&.$$

Here then $a_2 = 2a_1 - a + 2$, that is, by the method of increments, $a + 2\underset{\cdot}{a} + \underset{\cdot\cdot}{a} = 2a + 2\underset{\cdot}{a} - a + 2$, that is, $\underset{\cdot\cdot}{a} = 2 = 2\underset{\cdot}{n}$, and $\underset{\cdot}{a} = 2n + \text{A}$, and $a = \frac{2n\,n_{-1}}{2} + \text{A}n$; and when $\underset{\cdot}{n} = 1$, $\text{A} = 1$, therefore $a = n\,n_{-1} + n$.

Again, $b_2 = 2b_1 + a_1 - b$, and their increments $b + 2\underset{\cdot}{b} + \underset{\cdot\cdot}{b} = b + 2\underset{\cdot}{b} + a + \underset{\cdot}{a} - b$, and $\underset{\cdot\cdot}{b} = a + \underset{\cdot}{a}$, likewiſe $\underset{\cdot\cdot}{c} = b + \underset{\cdot}{b}$, $\underset{\cdot\cdot}{d} = c + \underset{\cdot}{c}$ and ſo on; that is, $\underset{\cdot\cdot}{b} = a_1$, $\underset{\cdot\cdot}{c} = b_1$, $\underset{\cdot\cdot}{d} = c_1$ &c; therefore $\underset{\cdot\cdot}{b} = n\,n_1 + n_1$, and $\underset{\cdot}{b} = \frac{n\,n_{-1}\,n_1}{3} + \frac{n\,n_1}{2}$, and $b = \frac{n\,n_{-2}\,n_{-1}\,n_1}{3\cdot4} + \frac{n\,n_{-1}\,n_1}{2\cdot3}$.

Again, $\underset{\cdot\cdot}{c} = \frac{n\,n_{-1}\,n_1\,n_2}{3\cdot4} + \frac{n\,n_1\,n_2}{2\cdot3}$, and $\underset{\cdot}{c} = \frac{n\,n_{-2}\,n_{-1}\,n_1\,n_2}{3\cdot4\cdot5} + \frac{n\,n_{-1}\,n_1\,n_2}{2\cdot3\cdot4}$, and $c = \frac{n_{-3}\ldots n_2}{3\ldots6} + \frac{n_{-2}\ldots n_2}{2\ldots5}$. In like manner $d = \frac{n_{-4}\ldots n_3}{3\ldots8} + \frac{n_{-3}\ldots n_3}{2\ldots7}$, and $e = \frac{n_{-5}\ldots n_4}{3\ldots10} + \frac{n_{-4}\ldots n_3}{2\ldots9}$, and ſo on. Then ſince $\underset{\cdot}{n} = 1$, and $n, n_{-1}, n_1, n_{-2}, n_2$ &c $= n, n-1, n+1, n-2, n+2$, &c, reſpectively. Therefore by reduction will be found $a = nn$, $b = \frac{nn\cdot\overline{nn-1}}{2\cdot2\cdot3}$, $c = \frac{nn\cdot\overline{nn-1}\cdot\overline{nn-4}}{3\cdot1\cdot2\cdot3\cdot4\cdot5}$, $d = \frac{nn\cdot\overline{nn-1}\cdot\overline{nn-4}\cdot\overline{nn-9}}{4\cdot1\ldots7}$, &c; *whence the* *ſquare of the cord of the* n^{th} *arch* $= nnx^2 - \frac{nn\cdot\overline{nn-1}}{2\cdot1\cdot2\cdot3}x^4 + \frac{nn\cdot\overline{nn-1}\cdot\overline{nn-4}}{3\cdot1\cdot2\cdot3\cdot4\cdot5}x^6 - \frac{nn\cdot\overline{nn-1}\cdot\overline{nn-4}\cdot\overline{nn-9}}{4\cdot1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}x^8$ &c; *or putting* $r =$ *radius*, □ cord n^{th} arch $= nnx^2 - \frac{\overline{nn-1}\cdot x^2}{2\cdot2\cdot3r^2}\mathrm{A} - \frac{2\cdot\overline{nn-4}\cdot x^2}{3\cdot4\cdot5\cdot rr}\mathrm{B} - \frac{3\cdot\overline{nn-3^2}}{4\cdot6\cdot7r^2}x^2\,\mathrm{C} - \frac{4\cdot\overline{n^2-4^2}x^2}{5\cdot8\cdot9r^2}\mathrm{D} - \frac{5\cdot\overline{n^2-5^2}\cdot x^2}{6\cdot10\cdot11\cdot r^2}\mathrm{E}$ &c; that is, □ cord of the n^{th} arch $= nnxx - \frac{\overline{nn-1}\cdot xx}{3\cdot4rr}\mathrm{A} - \frac{\overline{nn-4}\cdot xx}{5\cdot6rr}\mathrm{B} - \frac{\overline{nn-3^2}\cdot xx}{7\cdot8rr}\mathrm{C} - \frac{\overline{n^2-4^2}\cdot xx}{9\cdot10}\mathrm{D} - \frac{\overline{n^2-5^2}\cdot xx}{11\cdot12rr}\mathrm{E} -$ &c.

Where x^{2n} *will always be the laſt term,* n *being an integer.*

Cor. 1. *If* x *be the cord of an arch, radius* $= r$, *and* c *the cord of* n *times the arch; then* $\frac{x^n}{r^{n-1}} - \frac{nx^{n-2}}{r^{n-3}} + n\times\frac{\overline{n-3}\cdot x^{n-4}}{2r^{n-5}} - n\times\frac{\overline{n-4}\cdot\overline{n-5}\cdot x^{n-6}}{2\cdot3r^{n-7}} + n\times\frac{\overline{n-5}\cdot\overline{n-6}\cdot\overline{n-7}}{2\cdot3\cdot4r^{n-9}}x^{n-8} - n\times\frac{\overline{n-6}\cdot\overline{n-7}\cdot\overline{n-8}\cdot\overline{n-9}}{2\cdot3\cdot4\cdot5r^{n-11}}x^{n-10}$ &c $= \pm c$, *according as* $\frac{n+1}{2}$ *is odd or even, the ſeries continued to* $\frac{n+1}{2}$ *terms: that is,* $+c$ *when odd, and* $-c$ *when even.*

Or thus; $\sqrt{4rr-xx} : \mathrm{x}$ into $\frac{x^{n-1}}{r^{n-1}} - \frac{\overline{n-2}\cdot x^{n-3}}{r^{n-3}} + \frac{\overline{n-3}\cdot\overline{n-4}}{1\cdot 2r^{n-5}}x^{n-5} - \frac{\overline{n-4}\cdot\overline{n-5}\cdot\overline{n-6}}{1\cdot 2\cdot 3r^{n-7}}x^{n-7} + \frac{\overline{n-5}\cdot\overline{n-6}\cdot\overline{n-7}\cdot\overline{n-8}}{1\cdot 2\cdot 3\cdot 4r^{n-9}}x^{n-9} - \&c =$ $\pm\, c$, *according as* $\frac{1}{2}n$ *is odd or even, and the ſeries continued to* $\frac{n}{2}$ *terms.*

For the former ſeries is the root of $x^{2n} - 2nx^{2n-2} + 2n \times \frac{\overline{2n-3}}{2}x^{2n-4}$ &c, the ſquare of the cord, found above by the firſt method. And the latter ſeries is found, by dividing the ſeries $x^{2n} - 2nx^{2n-2}$ &c by $-xx + 4$ (or rather by $xx - 4$), and extracting the root of the quotient.

Cor. 2. *If* v *be the verſed ſine of an arch,* v *the verſed ſine of* n *times the arch, then will* $\frac{\overline{2v^n}}{2r^{n-1}} - \frac{2nr}{2v}\mathrm{A} - \frac{\overline{2n-3}\cdot r}{2\cdot 2v}\mathrm{B} - \frac{\overline{2n-4}\cdot\overline{2n-5}\cdot r}{3\cdot\overline{2n-3}\cdot 2v}\mathrm{C} - \frac{\overline{2n-6}\cdot\overline{2n-7}\cdot r}{4\cdot\overline{2n-4}\cdot 2v}\mathrm{D} - \frac{\overline{2n-8}\cdot\overline{2n-9}\cdot r}{5\cdot\overline{2n-5}\cdot 2v}\mathrm{E}\ \&c$ $= \pm\,\mathrm{v}$, *according as* n *is odd or even, the ſeries continued to* n *terms.*

Or $\mathrm{v} = n^2v - \frac{\overline{n^2-1}\cdot v}{2\cdot 3r}\mathrm{A} - \frac{\overline{n^2-2^2}\cdot v}{3\cdot 5r}\mathrm{B} - \frac{\overline{n^2-3^2}\cdot v}{4\cdot 7r}\mathrm{C} - \frac{\overline{n^2-4^2}\cdot v}{5\cdot 9r}\mathrm{D} - \&c.$ This appears by ſubſtituting $2rv$ for xx, and $2r\mathrm{v}$ for bb, to which they are equal, by Cor. 8 Prop. I.

Cor. 3: *The ſeries in* Cor. 1 *will give the ſine of n times the arch; putting x = ſine of the ſingle arch,* $\frac{1}{2}r$ *for r in the ſeries.*

PROP. XXX.

The cord of the ſupplement of an arch being given; to find the cord of the ſupplement of n times the arch.

Let radius $= r = 1$, $x =$ cord of the ſupplement, $\text{A} =$ arch; then by Cor. Prop. XXIII, $1 : x :: \text{cord} . \text{ſup.}\ n\text{A} : \text{cord} . \text{ſup.}\ \overline{n-1} . \text{A} + \text{cord} . \text{ſup.}\ \overline{n+1} . \text{A}$. Whence $\text{cord} . \text{ſup.}\ \overline{n+1} . \text{A} = x \times \text{cord} . \text{ſup.}\ n\text{A} - \text{cord} . \text{ſup.}\ \overline{n-1} . \text{A}$; that is, if the cord of the ſup. of any arch, in a ſeries thereof, be multiplied by x, and the preceding cord of the ſup. ſubtracted, you will have the next ſucceeding one. Hence

Cord ſup. $0\text{A} = 2$
$1\text{A} = x$
$2\text{A} = x^2 - 2$
$3\text{A} = x^3 - 3x$
$4\text{A} = x^4 - 4x^2 + 2$
$5\text{A} = x^5 - 5x^3 + 5x$
$6\text{A} = x^6 - 6x^4 + 9x^2 - 2$
&c.

And in general, $n\text{A} = x^n - ax^{n-2} + bx^{n-4} - cx^{n-6}$ &c; then

$$\overline{n+1} . \text{A} = x^{n+1} - \underset{1}{a}x^{n-1} + \underset{1}{b}x^{n-3} - \underset{1}{c}x^{n-5} \text{ \&c}$$

$$= \left\{ \begin{array}{l} x^{n+1} - ax^{n-1} + bx^{n-3} - cx^{n-5} \\ \qquad\quad - x^{n-1} \underset{-1}{+} ax^{n-3} \underset{-1}{-} bx^{n-5} \end{array} \right\} \text{\&c}$$

by conſtruction. Hence $\underset{1}{a} = a + 1$, and $\underset{:}{a} = 1 = n$, whence $a = n$.

Alſo $b_{1} = b + a_{-1}$, and $b = a_{-1} = n_{-1}$, whence $b = \frac{n_{-2}\,n_{-1}}{2} - 1$, for when $n = 4$, $b = 2$; therefore $b = \frac{n_{-3}\,n_{-2}}{2} + n_{-2} - 1 = \frac{n_{-3}\,n_{-2}}{2} + n_{-3} = \frac{n_{-3}\,n}{2}$.

Again, $c_{1} = c + b_{-1}$, and $c = b_{-1} = \frac{n_{-4}\,n_{-3}}{2} + n_{-4}$, and $c = \frac{n_{-3}\,n_{-4}\,n_{-5}}{2\cdot3} + \frac{n_{-4}\,n_{-5}}{2} = \frac{n\,n_{-4}\,n_{-5}}{1.2.3}$.

Again, $d = c_{-1} = \frac{n_{-4}\,n_{-5}\,n_{-6}}{2\cdot3} + \frac{n_{-5}\,n_{-6}}{2}$, and $d = \frac{n_{-4}\,n_{-5}\,n_{-6}\,n_{-7}}{2\cdot3\cdot4} + \frac{n_{-5}\,n_{-6}\,n_{-7}}{2\cdot3} = \frac{n\,n_{-5}\,n_{-6}\,n_{-7}}{1\cdot2\cdot3\cdot4}$. In like manner $e = \frac{n\,n_{-6}\,n_{-7}\,n_{-8}\,n_{-9}}{2\cdot3\cdot4\cdot5}$ &c; whence

The cord of the ſupplement of the n^{th} *arch is* $= x^{n} - nx^{n-2} + \frac{n\cdot\overline{n-3}}{1\cdot2}x^{n-4} - \frac{n\cdot\overline{n-4}\cdot\overline{n-5}}{1\cdot2\cdot3}x^{n-6}$ + &c $= \frac{x^{n}}{r^{n-1}} - \frac{n\cdot rr}{xx}A - \frac{\overline{n-3}\cdot rr}{2xx}B - \frac{\overline{n-4}\cdot\overline{n-5}\cdot rr}{3\cdot\overline{n-3}\cdot xx}C - \frac{\overline{n-6}\cdot\overline{n-7}\cdot rr}{4\cdot\overline{n-4}\cdot xx}D - \frac{\overline{n-8}\cdot\overline{n-9}\cdot rr}{5\cdot\overline{n-5}\cdot xx}E$ — &c, *continued to* $\frac{n+1}{2}$ *terms when n is odd, or to* $\frac{n+2}{2}$ *terms when even.*

Cor. *Hence, if radius* $= r$, $x =$ *coſine of an arch; then the coſine of the* n^{th} *arch* $= \frac{\overline{2x}^{n}}{2r^{n-1}} - \frac{nrr}{4.xx}\text{A} - \frac{\overline{n-3}.rr}{4.2xx}\text{B} - \frac{\overline{n-4}.\overline{n-5}.rr}{4.3.\overline{n-3}.xx}\text{C} - \frac{\overline{n-6}.\overline{n-7}.rr}{4.4.\overline{n-4}.xx}\text{D}$ — &c, *continued to* $\frac{n+1}{2}$ *terms when* n *is odd, or to* $\frac{1}{2}n + 1$ *terms when even.*

This follows from the foregoing ſeries, putting $\frac{1}{2}r$ for r.

PROP. XXXI.

Given the cord of an arch, and the cord of the ſupplement of twice the arch; to find the cord of $2m + 1$ *times the arch.*

Let radius $= r = 1$, arch $=$ A, cord of A $= y$, cord ſup. of 2A $= z$. Then by cor. Pr. XXIII, $1 : z ::$ cord of $\overline{2m+1}.\text{A}$: cord of $\overline{2m-1}.\text{A}$ + cord of $\overline{2m+3}.\text{A}$. Therefore cord of $\overline{2m+3}.\text{A} = z \times$ cord of $\overline{2m+1}.\text{A} -$ cord of $\overline{2m-1}.\text{A}$; that is, in a ſeries of theſe cords, if any one be multiplied by z, and the foregoing one ſubtracted, you will have the next following one. Whence

Cord $\text{A} = y$

$3\text{A} = z + 1 : \times y$

$5\text{A} = z^2 + z - 1 : \times y$

$7\text{A} = z^3 + z^2 - 2z - 1 : \times y$

$9\text{A} = z^4 + z^3 - 3z^2 - 2z + 1 : \times y$

$11\text{A} = z^5 + z^4 - 4z^3 - 3z^2 + 3z + 1 : \times y$

$13\text{A} = z^6 + z^5 - 5z^4 - 4z^3 + 6z^2 + 3z - 1 : \times y$

$15\text{A} = z^7 + z^6 - 6z^5 - 5z^4 + 10z^3 + 6z^2 - 4z - 1 : \times y$;

and in general, cord $\overline{2m+1}.\text{A} = z^m + z^{m-1} -$

$az^{m-2} - bz^{m-3} + cz^{m-4} + dz^{m-5} - ez^{m-6} - fz^{m-7}$ &c; then by conſtruction,

$$\left\{\begin{array}{l} z^{m+1} + z^{m} - az^{m-1} - bz^{m-2} + cz^{m-3} + dz^{m-4} - ez^{m-5} \\ \qquad\qquad\quad - z^{m-1} - z^{m-2} + \underset{-1}{a}z^{m-3} + \underset{-1}{b}z^{m-4} - \underset{-1}{c}z^{m-5} \end{array}\right.$$

$$= z^{m+1} + z^{m} - \underset{1}{a}z^{m-1} - \underset{1}{b}z^{m-2} + \underset{1}{c}z^{m-3} + \underset{1}{d}z^{m-4} - \underset{1}{e}z^{m-5} \text{ \&c,}$$

&c, = cord of $\overline{2m+3}$. A. Whence $\underset{1}{a} = a + 1$, and $\underset{1}{a} - a$ or $\underset{\cdot}{a} = 1 = \underset{\cdot}{m}$, whence $a = \underset{-1}{m}$, for when $m = 2$, $a = 1$.

Again, $\underset{1}{b} - b$ or $\underset{\cdot}{b} = 1 = \underset{\cdot}{m}$, and $b = \underset{-2}{m}$, for when $m = 3$, $b = 1$.

Alſo $\underset{1}{c} = c + \underset{-1}{a}$, and $\underset{1}{c} - c$ or $\underset{\cdot}{c} = \underset{-2}{m}$, whence $c = \dfrac{\underset{-3}{m}\,\underset{-2}{m}}{2}$.

Likewiſe $\underset{1}{d} - d$ or $\underset{\cdot}{d} = \underset{-1}{b} = \underset{-3}{m}$, and $d = \dfrac{\underset{-4}{m}\,\underset{-3}{m}}{2}$.

Alſo $\underset{\cdot}{e} = \underset{-1}{c}$, $= \dfrac{\underset{-4}{m}\,\underset{-3}{m}}{2}$, and $e = \dfrac{\underset{-5}{m}\,\underset{-4}{m}\,\underset{-3}{m}}{2 \cdot 3}$. Likewiſe $\underset{\cdot}{f} = \underset{-1}{d}$, and $f = \dfrac{\underset{-6}{m}\,\underset{-5}{m}\,\underset{-4}{m}}{2 \cdot 3}$, $g = \dfrac{\underset{-7}{m}\,\underset{-6}{m}\,\underset{-5}{m}\,\underset{-4}{m}}{2 \cdot 3 \cdot 4}$; $h = \dfrac{\underset{-8}{m}\,\underset{-7}{m}\,\underset{-6}{m}\,\underset{-5}{m}}{2 \cdot 3 \cdot 4}$; $i = \dfrac{\underset{-9}{m} \ldots \underset{-5}{m}}{2.3.4.5}$, $k = \dfrac{\underset{-10}{m} \ldots \underset{-6}{m}}{2.3.4.5}$ &c; whence the

Cord of the $\overline{2m+1}$. *arch* $= \underline{y \times \text{into}}\ z^{m} \underline{+ z^{m-1}} - \dfrac{m-1}{1}z^{m-2} - \dfrac{m-2}{1}z^{m-3} + \dfrac{m-2 \,.\, m-3}{2}z^{m-4}$

$$+\frac{\overline{m-3}\cdot\overline{m-4}}{2}z^{m-5}-\frac{\overline{m-3}\cdot\overline{m-4}\cdot\overline{m-5}}{2\cdot 3}z^{m-6}$$
$$-\frac{\overline{m-4}\cdot\overline{m-5}\cdot\overline{m-6}}{2\cdot 3}z^{m-7}+\frac{\overline{m-4}\ldots\overline{m-7}}{2\cdot 3\cdot 4}z^{m-8}$$
$$+\frac{\overline{m-5}\ldots\overline{m-8}}{2\cdot 3\cdot 4}z^{m-9}-\frac{\overline{m-5}\ldots\overline{m-9}}{2\cdot 3\cdot 4\cdot 5}z^{m-10}$$
$$-\frac{\overline{m-6}\ldots\overline{m-10}}{2\cdot 3\cdot 4\cdot 5}z^{m-11}$$ &c, continued to $m+1$ terms.

PROP. XXXII.

Given the cord of the ſupplement of an arch, and the cord of the ſupplement of twice the arch; to find the cord of the ſupplement of $\overline{2m+1}$ *times the arch.*

Let radius $= 1$, arch $= A$, cord ſup. $A = x$, cord ſup. $2A = z$; then as in the laſt prop. cord . ſup. $\overline{2m+3}\cdot A = z \times$ cord . ſup. $\overline{2m+1}\cdot A -$ cord . ſup. $\overline{2m-1}\cdot A$; that is, if, in a ſeries, any one be multiplied by z, and the foregoing ſubtracted, you will have the next following one. Hence cord . ſup.

of $A = x$
$3A = \overline{z - 1} \times x$
$5A = \overline{z^2 - z - 1} \times x$
$7A = \overline{z^3 - z^2 - 2z + 1} \times x$
$9A = \overline{z^4 - z^3 - 3z^2 + 2z + 1} \times x$
$11A = \overline{z^5 - z^4 - 4z^3 + 3z^2 + 3z - 1} \times x$ &c.

And in general, cord . ſup. $\overline{2m+1}\cdot A = z^m - z^{m-1} - az^{m-2} + bz^{m-3} + cz^{m-4} - dz^{m-5} - ez^{m-6}$ &c: therefore by conſtruction

$$\left.\begin{matrix} z^{m+1} - z^m - az^{m-1} + bz^{m-2} + cz^{m-3} \\ -1z^{m-1} + 1z^{m-2} \underset{-1}{+} az^{m-3} \end{matrix}\right\}\text{&c} =$$

$$z^{m+1} - z^m - \underset{1}{a}z^{m-1} + \underset{1}{b}z^{m-2} + \underset{1}{c}z^{m-3} =$$ cord . ſup. $\overline{2m+3}\cdot A$; hence $\underset{1}{a} = a + 1$, $\underset{1}{b} = b + 1$, $\underset{1}{c} = c + \underset{-1}{a}$, $\underset{1}{d} =$

$\underset{1}{d} = d + \underset{-1}{b}$ &c, all as in the laſt prop. and therefore the coefficients a, b, c, &c, will be the very ſame, whence the

$$Cord . ſup. \overline{2m+1} . A = x \times \text{into } z^{m} - z^{m-1} - \frac{\overline{m-1}}{1} z^{m-2} + \frac{\overline{m-2}}{1} z^{m-3} + \frac{\overline{m-2} . \overline{m-3}}{2} z^{m-4} - \frac{\overline{m-3} . \overline{m-4}}{2} z^{m-5} - \frac{\overline{m-3} . \overline{m-4} . \overline{m-5}}{2 . 3} z^{m-6} + \frac{\overline{m-4} . \overline{m-5} . \overline{m-6}}{2 . 3} z^{m-7} + \frac{\overline{m-4} \ldots . \overline{m-7}}{2 . 3 . 4} z^{m-8} \text{ \&c.}$$

Cor. 1. *If rad.* $= 1$, $z =$ *cord . ſup. of* $\frac{1}{2m+1}$ *part of the circumference, then* $0 = z^{m} - z^{m-1} - \frac{\overline{m-1}}{1} z^{m-2} + \frac{\overline{m-2}}{1} z^{m-3} + \frac{\overline{m-2} . \overline{m-3}}{2} z^{m-4}$ &c.

For if $\overline{2m+1} \times A =$ ſemicircumference, then the cord of the ſup. and the general ſeries is $= 0$, and may be divided by x. And in that ſeries $z =$ cord . ſup. 2A, or of the $\overline{2m+1}$th part of the circumference.

PROP. XXXIII.

The tangent of an arch A *being given; to find the ſine and coſine of n times the arch.*

Let $t = \overline{\text{tan. A}}$; then by Prop. XXIV, S. nA $= \frac{nx^{n-1}y}{r^{n-1}} - \frac{\overline{n-1} . \overline{n-2} . yy}{2 . 3 . xx}$A &c: but $\frac{y}{x} = \frac{t}{r}$, and $x = \frac{rr}{\sqrt{rr+tt}}$, therefore $\frac{nx^{n-1}y}{r^{n-1}}$ or $\frac{nx^{n}ry}{r^{n}x} = \frac{nx^{n}t}{r^{n}} = \frac{nr^{n}t}{\overline{rr+tt}|^{\frac{n}{2}}}$, whence S. nA $= \frac{nr^{n}t}{\overline{rr+tt}|^{\frac{n}{2}}} + \frac{\overline{n-1} . \overline{n-2}\, tt}{2 . 3\, rr}$ A &c.

Likewiſe by Prop. XXVI, Coſ. $n\text{A} = \frac{x^n}{r^{n-1}} - \frac{n.\overline{n-1}.yy}{2xx}\text{A}$ &c; that is $= \frac{r^{n+1}}{\overline{rr+tt}|^{\frac{n}{2}}} - \frac{n.\overline{n-1}.tt}{2rr}\text{A}$ &c; whence

$$S.n\text{A} = \frac{nr^nt}{\overline{rr+tt}|^{\frac{n}{2}}} - \frac{\overline{n-1}.\overline{n-2}.tt}{2.3rr}\text{A} - \frac{\overline{n-3}.\overline{n-4}.tt}{4.5rr}\text{B}$$
$$- \frac{\overline{n-5}.\overline{n-6}.tt}{6.7rr}\text{C} - \frac{\overline{n-7}.\overline{n-8}.tt}{8.9rr}\text{D} - \&c,$$ and

$$Coſ.\,n\text{A} = \frac{r^{n+1}}{\overline{rr+tt}|^{\frac{n}{2}}} - \frac{n.\overline{n-1}.tt}{2rr}\text{A} - \frac{\overline{n-2}.\overline{n-3}.tt}{3.4rr}\text{B}$$
$$- \frac{\overline{n-4}.\overline{n-5}.tt}{5.6rr}\text{C} - \frac{\overline{n-6}.\overline{n-7}.tt}{7.8rr}\text{D}\ \&c.$$

Cor. 1. Hence ſince $t = \frac{ry}{x}$, therefore *tan.* $n\text{A} =$

$$\frac{nr^nt - \frac{n.\overline{n-1}.\overline{n-2}}{2.3}r^{n-2}t^3 + \frac{n\ldots\ldots\overline{n-4}}{2.3.4.5}r^{n-4}t^5\ \&c,}{r^n - \frac{n.\overline{n-1}}{1.2}r^{n-2}tt + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4}r^{n-4}t^4\ \&c.}$$

Cor. 2. If $ſ =$ ſecant of A, and $t =$ tan. A, then

$$ſecant\ n\text{A} = \frac{ſ^n}{r^{n-1} - \frac{n.\overline{n-1}}{2}r^{n-3}tt + \frac{n\ldots\overline{n-3}}{1\ldots4}r^{n-5}t^4 - \&c;}$$

for $\sqrt{rr+tt} = ſ$, and $\frac{rad.^2}{coſine} =$ ſecant.

SCHOLIUM.

Tho' the principal uſe of the foregoing propoſitions is when n is an integer number, in which caſe the ſeries will always terminate. Yet moſt of theſe ſeries will be equally true when n is any fractional number, if they do but converge. For the inveſtigation of any of theſe propoſitions does not ſuppoſe that n is always an integer, it only ſuppoſes that n is 1.

SECT.

SECT. V.

Angular ſections, the inſcription of polygons, the properties of the cords or ſubtenſes of arches.

PROP. XXXIV.

FIG. 7.

If any arch of a circle as AR *be given, and there be taken the arch* AD $= \frac{1}{n}$ *the arch* AR; *n being any integer number; and if the whole circumference of the circle be divided into n equal parts at the points* D, E, F, G, H, *&c, beginning at* D: *Then I ſay any of the arches* AD, AE, ADF, AEG, AEH, *&c, repeated n times, from* A, *will always end at* R.

Call the whole circumference C; then by conſtruction, $n \times AD = AR$, alſo $n \times DE = C$, and $n \times DEF = 2C$, and $n \times DEG = 3C$, $n \times DFH = 4C$, *&c*; therefore

$$\begin{array}{lllll} & n \times AE & \text{or } n \times AD + n \times DE & = AR + & C \\ \text{and } & n \times AF & \text{or } n \times AD + n \times DF & = AR + & 2C \\ \text{and } & n \times AEG & \text{or } n \times AD + n \times DEG & = AR + & 3C \\ & n \times AFH & \text{or } n \times AD + n \times EFH & = AR + & 4C \ \&c. \end{array}$$

Therefore $n \times AE$, $n \times AF$, $n \times AEG$, *&c*, $= 1, 2, 3$, *&c*, entire circumferences + the arch AR. And ſince any number of circumferences will end at the point A where they began; conſequently by the addition of the arch AR, they will neceſſarily all end at R.

Cor. *If the cords* AR, BR, AD, AE, AF, *&c, are drawn; then the cord* AR = *cord of n times* AD, *or of n times* AE, *or of n times* AEF, *or of n times* AEG, &c; *and* BR *is the cord of the ſupplement of each of theſe.*

PROP. XXXV.

FIG. 7. *To divide an arch* AR *into any number of equal parts, as n.*

Let the radius = 1, b = cord of the ſup. of AR, or the cord of BR, x = BD the cord ſup. firſt part AD. Then by Prop. XXX, $x^n - nx^{n-2} + \frac{n.\overline{n-3}}{2}x^{n-4}$ &c $= \pm b$, according as the arch AR is leſſer or greater than a ſemicircle. Therefore x, the greateſt root of this equation, being extracted, will give BD, the cord of the ſupplement of the arch required.

EXAMPLE.

If the arch AR be leſs than a ſemicircle, and is to be divided into 7 equal parts; then $n = 7$, and you will have $x^7 - 7x^5 + 14x^3 - 7x = b$, whoſe greateſt root x is BD, the cord of the ſupplement of AD required.

Otherwiſe.

Let rad. = 1, cord AR = b, cord AD = x, then, by Prop. XXIX, you will have $x^{2n} - 2nx^{2n-2} + 2n \times \frac{\overline{2n-3}}{2}x^{2n-3}$ &c to n terms $= \pm bb$, according as n is odd or even. Then extract the leaſt root xx of this equation, and then its ſquare root x will be $\doteq$ AD, the cord of the firſt of the equal parts. *Or thus,* put $nnxx - \frac{nn.\overline{nn-1}}{2.1.2.3}x^4 + \frac{nn.\overline{nn-1}.\overline{nn-4}}{3.1.2.3.4.5}x^6$ &c $\doteq bb$, &c.

EXAMPLE.

If AR was to be divided into 5 equal parts, then $n = 5$, and $x^{10} - 10x^8 + 35x^6 - 50x^4 + 25x^2 \mp bb$, whoſe leaſt root xx is the ſquare of AD, and $x \doteq$ AD, the cord of $\frac{1}{n}$ part of the arch AR.

Cor.

FIG. 7. *Cor. If you continue to extract all the roots z, you will get the values of all the cords* BE, BF, BG, &c, *by the former method; or of the cords* AE, AF, AG, &c, *by the latter method; as appears by cor. Prop. XXXII.*

SCHOLIUM.

If n is not a prime number, but compounded of two or more primes; it will be eaſier to divide the arch into a number of parts denoted by one of theſe compounding numbers, and then the firſt of theſe parts into as many parts as is denoted by another of them, and ſo on to the laſt.

PROP. XXXVI.

To inſcribe any regular polygon in a circle.

Let the number of its ſides be $= 2m + 1$, an odd number. Radius $= 1$, $z =$ the unknown cord BD, or the cord of the ſupplement of one part AD. Then by cor. Pr. XXXII, $z^m - z^{m-1} - \frac{m-1}{1}z^{m-2} + \frac{m-2}{1}z^{m-3}$ &c $= 0$; *therefore the greateſt root* z *will give* BD, *and* $\sqrt{4 - zz} =$ *ſide of the polygon* AD, *required.* 8.

EXAMPLE.

In a heptagon, $2m + 1 = 7$, or $m = 3$; then by the former equation $z^3 - z^2 - 2z + 1 = 0$, and the greateſt root z gives the value of BE, and $\sqrt{4 - zz} =$ AE, the ſide of the heptagon. 9.

Otherwiſe.

Let $n =$ number of ſides, rad. $= 1$, $x =$ cord of the ſup. of the n^{th} part of the circle. Then ſince the

FIG. 9. the cord ſubtending the whole circle is 0; therefore by Prop. XXVIII, $yx^{n-1} - \frac{n-2}{1}yx^{n-3}$ &c $= 0$; therefore, dividing by y, you will have $x^{n-1} - \frac{n-2}{1}x^{n-3} + \frac{n-3 \cdot n-4}{1 \cdot 2}x^{n-5}$ &c $= 0$; *extract the greateſt root* x *of this equation, then* $\sqrt{4-xx} =$ *ſide of the polygon.*

EXAMPLE.

In a pentagon $n = 5$, then $x^4 - 3x^2 + 1 = 0$, and $x^2 = \frac{3+\sqrt{5}}{2}$, the greateſt root, and $\sqrt{4-xx} = \sqrt{\frac{5-\sqrt{5}}{2}}$ the ſide of the pentagon.

SCHOLIUM.

When the number of ſides is not prime; it will be eaſier to divide the circumference into a number of parts denoted by one of the primes that compoſe it; and then one of theſe parts into as many parts as is denoted by another of the compounding prime numbers, and ſo on till all the primes which compoſe the given number be exhauſted.

PROP. XXXVII.

10. *If the arch* DAF *of a circle, whoſe diameter is* AE, *be divided into an even number of equal parts at* C, B, A, H, G, *&c; and there be drawn the lines* BH, CG, DF, &c, *perpendicular to the diameter* AE; *and likewiſe the cords* AB, BE: *I ſay, the ſum of the lines,* BH, CG, &c, + *half the laſt, that is,* $BH + CG + DO = \frac{AO \times EB}{AB}$.

For draw the lines CH, DG, &c; then all the triangles BAK, KHL, LCM, MGN, NDO, &c, are ſimi-

lar. Therefore $BK : KA :: HK : KL :: CM : ML$ FIG. 10.
$:: GM : MN :: DO : ON$, &c; therefore $BK : AK :: BK + KH + CM + GM + DO : AK + KL + LM + MN + NO :: BH + CG + DO : AO$. But $EB : AB :: BK : AK :: BH + CG + DO : AO$; therefore $AB \times \overline{BH + CG + DO} = AO \times EB$.

Cor. 1. *If the circle* ADG *be divided into an even number of equal parts at* A, B, C, D, E, F, &c; *and all the lines* BH, CG, DF, &c, *be drawn; and alſo* AB, BE. *Then the ſum of all the lines* $BH + CG + DF$ &c $= \frac{AE \times EB}{AB}$.

For here the baſe of the ſegment is O, and the points D, O, F, E coincide.

Cor. 2. *If a ſemicircle* ABE *be divided into any number of equal parts, as* n, *at the points* B, C, D, &c; *and the ſines* BK, CM, DO, &c, *be drawn; then I ſay, the ſum of all the ſines,* $BK + CM + DO$ &c $=$ $\frac{AE \times EK}{2BK}$.

For by ſimilar triangles $\frac{EB}{AB} = \frac{EK}{BK}$; and taking half of the quantities in Cor. 1, $BK + CM + DO = \frac{AE \times EB}{2AB} = \frac{AE \times EK}{2BK}$. Hence

Cor. 3. *Let radius* $= r$, $\frac{180}{n} = A$, n *being any integer.* $s =$ *ſin.* A, $c =$ *coſ.* A; *then the ſum of all the ſines of* A, 2A, 3A, 4A, &c, *in the ſemicircle, is* $= \frac{r + c}{s} r$.

FIG. 10. Cor. 4. *Hence the area of the curve, made by erecting all the ſines perpendicular to a line equal to the ſemicircumference, or as it is commonly called, the area of the figure of ſines, is* $= 2rr$.

For if n be infinite, and A be $= 1 = s$, then $c = r$, and the ſum of all the ſines or area of the curve $= \frac{2rr}{s} = 2rr$.

PROP. XXXVIII.

10. *If the circumference of the circle* ADO *be divided into an even number of equal parts, at the points* A, B, C, D, &c; *and the parallel cords* AB, CH, DG, &c, *be drawn; I ſay, the ſum of all the cords* $AB + CH + DG + EF$ &c $= \frac{AE^2}{AB}$.

For by ſimilar triangles, AE : AB :: AB : AK :: HL : KL :: CL : ML :: GN : MN :: DN : NO :: FE : EO; and by compoſition AE : AB :: AB + CH + DG + FE : AE.

Cor. 1. *If a circle is divided into an even number of equal parts, and* A *be one of the parts. Then the ſum of all the cords of* 2A, 4A, 6A, 8A, &c, *is to the ſum of all the cords of* A, 3A, 5A, 7A, &c, :: *as* EB : *to* EA.

For BH, CG, DF, &c, are the cords of 2A, 4A, 6A. And BA, CH, DG, &c, are the cords of 1A, 3A, 5A, &c. And the firſt ſum, is to the laſt, as $\frac{AE \times EB}{AB}$ to $\frac{AE^2}{AB}$, or as EB : to AE.

Cor. 2. *If a ſemicircle is divided into any even number of equal parts, as* n; *and if* A *be one of theſe parts, and* $s =$ *ſin.* A, *radius* $= r$; *then the ſum of the ſines of* A, 3A, 5A, 7A, &c, $= \frac{rr}{s} =$ *coſec.* A.

This follows from this prop. taking half the cord for the ſine of half the arch. FIG. 10.

Cor. 3. *If a ſemicircle is divided into an even number of equal parts, and* A *be one of theſe parts, and rad.* $= r$, *ſin.* A $= s$, *coſ.* A $= c$; *then the ſum of all the even ſines in the ſemicircle, that is, the ſum of the ſines of* 2A, 4A, 6A, 8A, &c, $= \frac{cr}{s} =$ *cotan.* A.

For by Cor. 3. Pr. 37, ſum of the ſines of } A, 2A, 3A, 4A, &c, $= \frac{r+c}{s}r$.

And by the laſt cor. ſum of the ſines of } A, 3A, 5A, &c, $= \frac{rr}{s}$.

Therefore, dif. = 2A 4A, 6A, &c, $= \frac{cr}{s}$.

PROP. XXXIX.

If the circumference of a circle, whoſe diameter is AB *and center* C, *be divided into any odd number of equal parts, as n, at the points* D, E, F, G, H, I, K; *and if, from any point* A, *there be drawn the cords* AD, AE, AF, AG, AH, &c; *and if there be taken the arch* AR $=$ *n* × *arch* AD, *and the cord* AR *drawn*; *I ſay, the product of all the cords* AD × AE × AF × AG × AH × AI × AK $=$ AR × CB^{n-1}. 71

1. Let the line QAP be ſuppoſed to revolve about the point A, the common origin of all the arches; then will the part AP paſs thro' the circumference of the circle AEBHA. And if the line ſtill continues to revolve about A, then the other end AQ will alſo paſs thro' all the points of the circumference AEBHA. And if the line ſtill revolves forward, then the part AP paſſes again thro' the circle AEBHA, &c; and ſo on alternately. Now if from A to any point of interſection as R, the cord AR be drawn,

FIG. 7. drawn, this cord AR will be affirmative in going the firſt time about the circle; but negative, the ſecond time about; becauſe then AR will lie from A towards Q, in the revolving line, which is the contrary way to what it laid before; likewiſe in going the third time about, AR will be affirmative, and the fourth time negative; and ſo on alternately. Therefore in general, let C = circumference, A = any arch leſs than C; then the cords of A, 2C + A, 4C + A, &c, or of any even number of C's + A, will be affirmative. And the cords of C + A, 3C + A, 5C + A, &c, or of any odd number of C's + A, will be negative.

2. Let AR $= c$, $x =$ cord of the ſingle arch. Now ſince n times the arch AD, AE, AF, &c, all end at R (by Prop. XXXIV,) and therefore have the ſame cord AR; conſequently, by Cor. 1 Pr. XXIX, we ſhall have

Caſe 1. $x^n - nx^{n-2}$ &c $= + c$, the cords of A,
2C + A, 4C + A &c.
or the cords of $n \times$ AD, $n \times$ AF, $n \times$ AH &c.
Caſe 2. $x^n - nx^{n-2}$ &c $= - c$, the cords of
C + A, 3C + A, 5C + A &c.
or the cords of $n \times$ AE, $n \times$ AG, $n \times$ AEH &c.

And in caſe 1st, the root AD, being affirmative, the others AF, AH &c, are affirmative for the ſame reaſon.

And in caſe 2d, the root AE is affirmative, and likewiſe the roots AG, AI. But it is ſhewn in algebra, that if the ſigns of all the even powers in an equation be changed, the affirmative roots are changed into negatives, and the contrary. And in caſe 2d, all the even powers of x are wanting, except c; therefore in caſe 2d, changing $-c$ into $+c$, we have $x^n - nx^{n-2}$ &c $= + c$, in which equation, AE, AG, AI &c, are negative roots. But this equation is now the ſame as Caſe 1; therefore, in general, in the equation $x^n - nx^{n-2}$ &c $= c$, the affirmative roots are AD, AF, AH &c, and the negative ones AE, AG, AI &c. FIG. 7.

3. It is ſhewn in algebra, that in any equation, as $x^n - nx^{n-2}$ &c $= c$, or rather $x^n - nr^2x^{n-2}$ &c $= cr^{n-1}$, the abſolute number is equal to the product of all the roots; therefore AD × AE × AF × AG &c $= cr^{n-1}$.

Cor. 1. *The ſum of all the odd cords is equal to the ſum of all the even ones,* AD + AF + AH + AK = AE + AG + AI.

For it is ſhewn in algebra, that the coefficient of the ſecond term is = ſum of all the roots; and here the ſecond term is wanting, and therefore AD — AE + AF — AG &c = 0.

Cor. 2. *If the ſemicircumference be divided into any odd number of equal parts, as n; whereof the two firſt are* BF, *the two next* EF, *the two next* ED, *&c; and if the cords* AF, AE, AD, *&c; be drawn: then the product of all theſe cords,* AF × AE × AD $= \text{CB}^{\frac{n-1}{2}}$. 11.

For if AD = AK, then R and G fall in B, and AE = AI, AF = AH, &c; therefore by this prop. $\text{AD}^2 \times \text{AE}^2 \times \text{AF}^2 \times \text{AB} = \text{AB} \times r^{n-1}$. 7.

Cor. 3. *The ſame things ſuppoſed as in the laſt cor. I ſay,* AF — AE + AD *&c* = CB.

For by Cor. 1, 2AD + 2AF = 2AE + AB.

Cor. 4. *If arch* BR $= n \times$ BG, *and the cord* AR *drawn, and the reſt be done as is ſuppoſed in the prop. then the product of all the cords* AD × AE × AF *&c* $= \text{AR} \times \text{CB}^{n-1}$, *and* AD + AF + AH + AK = AE + AG + AI.

For

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FIG. 7. For it will appear with a little conſideration, that $AD + BG =$ half of $DE = \frac{c}{2n}$, or $n \times AD + n \times BG = \frac{1}{2}c$. But $n \times BG = BR$, therefore $n \times AD = \frac{1}{2}c - BR = AR$, therefore this cor. comes to the caſe of this propoſition.

Cor. 5. *If a quadrant be divided into any odd number of equal parts, as n; and* A *be one of theſe parts, radius = r. Then the product of all the ſines, of* 1A, 3A, 5A, 7A, *&c, in the quadrant* = $2 \times \overline{\frac{r}{2}}\big|^{\frac{n+1}{2}}$.

This follows from Cor. 2, taking half the cord for the ſine of half the arch, and half radius for CB, and multiplying both ſides by *r*.

PROP. XL.

9. *If the circumference of a circle, whoſe center is* C, *be divided into any odd number of equal parts, as n, at the points* A, E, F, G, H, I, K, *and from any one, as* A, *to all the reſt, there be drawn the cords* AE, AF, AG, *&c. I ſay the product of all the cords,* $AE \times AF \times AG \times AH \times AI \times AK = n \times AC^{n-1}$.

For let the arches AD, AR be infinitely ſmall, and they will coincide with their cords; then the cord AD, is to the cord AR, as 1 to *n*, and with this ratio the cords AD, AR vaniſh when A and D coincide, and therefore by Prop. XXXVII, $1 \times AE \times AF \times AG$ &c $= n \times CB^{n-1}$.

Otherwiſe.

If AD be = 0, then *n* times the arch AE, AF, AG, &c, always ends at A, and therefore *c*, the cord

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FIG. 7. of theſe multiple arches is = 0, therefore if s = cord of the ſingle arch, we ſhall have (by Cor. 2 Prop. xxv,) $ns - \frac{n.\overline{nn-1}}{24r^2}s^3$ &c = 0, wherein one root s is = 0, and the equation being divided, we have $n - \frac{n^3-n}{24rr}s^2$ &c = 0, wherein the abſolute number is n or rather nr^{n-1}, which therefore is equal to the product of all the roots, whoſe number is now $n-1$.

9. Cor. 1. *If the ſemicircumference* AEB *be divided into any odd number of equal parts, as n; whereof the two firſt is* AE, *the two next* EF, *the two next* FG, *&c. And the cords* AE, AF, AG, *&c, be drawn: I ſay, the product of all the cords* AE × AF × AG *&c* = $CB^{\frac{n-1}{2}} \times \sqrt{n}$.

For AE = AK, AF = AI, &c; therefore $AE^2 \times AF^2 \times AG^2 = n \times AC^{n-1}$.

Cor. 2. *If the quadrant of a circle be divided into any odd number of equal parts, as n, and* A *be one of them; then the product of all the coſines of* A, 3A, 5A, 7A, *&c,* = $\overline{\tfrac{r}{2}}\Big|^{\frac{n-1}{2}} \times \sqrt{n}$.

This follows from the laſt cor. becauſe ½AG, ½AF, &c, are the coſines of ½BG, ½BF, &c, or of A, 3A, &c.

Cor. 3. *Hence alſo, the ſame things ſuppoſed as in this prop. the ſum of the ſquares of the cords* $AE^2 + AF^2 + AG^2 + AH^2 + AI^2 + AK^2 = 2n \times CB^2$.

For in the equation $ns - \frac{n^3-n}{24rr}s^3$ &c = 0, or rather by Cor. 1 Prop. xxix, $x^n - nx^{n-2}$ &c = 0, one root $x = 0$, which being divided, $x^{n-1} - nx^{n-3}$ &c = 0, ſince here the index decreaſes by 2, there-

fore

FIG. 9. fore xx is the root of the equation, and n or nr^2 the coefficient of the ſecond term is the ſum of the roots, that is, of the ſquares of AE, AF, AG; which doubled gives the ſum of all the ſquares.

PROP. XLI.

12. *If the circumference of a circle, whoſe center is* C, *and diameter* AB, *be divided into any odd number of equal parts, at the points* D, E, F, G, H, I; *and from any point* A, *there be drawn the cords* AD, AE, AF, &c. *And the arch* AR *be taken* $= n \times$ *arch* AD, *and the cord* AR *drawn; then the product of all the cords* $AD \times AE \times AF \times AG \times AH \times AI = AR \times CB^{n-1}$.

Let cord $AR = c$, and $x =$ cord of the ſingle arch, then by Prop. XXIX we have $- x^{2n} + 2nx^{2n-2}$ &c $= cc$. But xx or the roots of this equation are AD^2, AE^2, AF^2, &c; and therefore their product $AD^2 \times AE^2$ &c $= cc$ or rather ccr^{2n-2}, and extracting the root, $AD \times AE \times$ &c $= cr^{n-1}$.

Cor. 1. *The ſum of the ſquares of all the cords* $AD^2 + AE^2 + AF^2 + AG^2 + AH^2 + AI^2 = 2n \times CB^2$.

For the coefficient of the ſecond term is the ſum of all the roots $= 2n$ or $2nr^2$.

13. Cor. 2. *If the ſemicircle* AEB *be divided into an even number of equal parts, as* n; *and the odd cords* AD, AE, AF *drawn; I ſay their product* $AD \times AE \times AF = CA^{\frac{n}{2}} \times \sqrt{2}$.

12. For in this caſe, $AD = AI$, $AE = AH$, and $AF = AG$, and $AR = 2CB$, therefore $AD^2 \times AE^2 \times AF^2$ $AR \times CB^{n-1} = 2CB^n$.

PROP. XLII.

If the circumference of a circle, whoſe center is C, *be divided into any even number of equal parts, at* A, D, E, *&c, and from one of them, as* A, *there be drawn to all the reſt the cords* AD, AE, AB, AG, AH; *I ſay the product of all the cords,* AD × AE × AB × AG × AH = $n \times CB^{n-1}$. FIG 8.

For $n \times AD$, $n \times AE$, &c, all end at A, and their cords = 0, therefore by Cor. 1 Prop. XXIX, $x^{n-1} - \overline{n-2}.x^{n-3}$ &c... $+ \frac{1}{2}nx : \times \sqrt{4-xx} = 0$, and one root $x = 0$, and dividing the equation by it and the correſpondent cord of the ſupplement $\sqrt{4-xx}$ (which here is AB); and we ſhall have $\pm x^{n-2} \mp \overline{n-2}.x^{n-4} \ldots + \frac{1}{2}n = 0$, where $\frac{1}{2}n$ or $\frac{1}{2}nr^{n-2}$ is the rectangle of the remaining roots AD, AE, AG, AH, which multiply by $2r$, and you have AD × AE × AB &c $= nr^{n-1}$.

This prop. is alſo evident from the laſt prop. by ſuppoſing AD, AR infinitely ſmall, and to vaniſh in the ratio of 1 to n.

PROP. XLIII.

If the circumference of a circle AFHA, *whoſe diameter is* AB *and center* C, *be divided into any even number of equal parts, as* n, *at the points* D, E, F, G, H, I; *and from any point* B *in the circumference, there be drawn the cords* BD, BE, BF, *&c, and if the arch* AR *be taken* = n × *arch* AD, *and the cord* BR *drawn; I ſay,* 14. 15.

The product of the ſquares of the odd cords $BD^2 \times BF^2 \times BH^2 = \overline{BA + BR} \times CB^{n-1}$, *when* $\frac{1}{2}n$ *is an odd number; and* $= \overline{BA - BR} \times CB^{n-1}$, *when* $\frac{1}{2}n$ *is an even number.*

FIG. 14. 15. *And the product of the ſquares of the even cords* $BE^2 \times BG^2 \times BI^2 = \overline{BA - BR} \times CB^{n-1}$, *when* $\frac{1}{2}n$ *is odd, and* $= BA + BR \times CB^{n-1}$, *when* $\frac{1}{2}n$ *is even.*

For let A be the common origin of the arches; and let the line PQ revolve about B, beginning at A, and going through the ſemicircumference AEB, in which any cord as BR being drawn, will be affirmative. But after the point of interſection paſſes through B, the other part BQ paſſes through the whole circumference, in which any cord BR will be negative; then the part PB paſſes through the next circumference from B, and any cord drawn in it will be affirmative; and ſo on alternately. Therefore if C = circumference, A = arch AR leſs than $\frac{1}{2}$C, then will the cords of A, 2C + A, 4C + A, &c, be affirmative, and the cords of C + A, 3C + A, 5C + A, &c, negative. Put a = cord of BR, then

CASE I.

In the arches AD, AF, AH, whoſe ſup. cords are BD, BF, BH; we have $n \times AD = A$, $n \times AF = 2C + A$, $n \times AFH = 4C + A$, &c; wherefore the cords of $n \times AD$, $n \times AF$, $n \times AFH$ are affirmative. Whence by Prop. XXX,

$$x^n - nx^{n-2} \text{ \&c } - 2 = +a, \text{ when } \tfrac{1}{2}n \text{ is odd,}$$

and $x^n - nx^{n-2}$ &c $+ 2 = +a$, when $\frac{1}{2}n$ is even, that is $x^n - nx^{n-2}$ &c $= a \pm 2$, according as $\frac{1}{2}n$ is odd or even. In which equation the ſquares of BD, BF, BH, the odd cords, are the roots; therefore their product is $2 \pm a$, or rather $\overline{2r \pm a} \times r^{n-1}$, according as $\frac{1}{2}n$ is odd or even.

CASE

CASE II.

In the arches AE, AEG, AEI, whoſe ſup. cords are BE, BG, BI; we have $n \times AE = C + A$, $n \times AEG = 3C + A$, $n \times AEI = 5C + A$. Therefore the cords of $n \times AE$, $n \times AEG$, $n \times AEI$, are negative, whence by Prop. xxx, FIG. 14. 15.

$$x^n - nx^{n-2} \text{ \&c} - 2 = -a, \text{ when } \tfrac{1}{2}n \text{ is odd,}$$

$$\text{and } x^n - nx^{n-2} \text{ \&c} + 2 = -a, \text{ when } \tfrac{1}{2}n \text{ is even;}$$

that is, $x^n - nx^{n-2}$ &c $+ a \mp 2 = 0$, as $\frac{1}{2}n$ is odd or even. Wherein the ſquares of BE, BG, BI, the even cords are the roots, and therefore their product is $a \mp 2$ or rather $\overline{2r \mp a} \times r^{n-1}$, according as $\frac{1}{2}n$ is odd or even.

Cor. 1. *The ſame things ſuppoſed; the ſum of the ſquares of the odd cords, as alſo the ſum of the ſquares of the even cords,* $BD^2 + BF^2 + BH^2$ or $BE^2 + BG^2 + BI^2 = n \times CB^2$. 14.

For the coefficient of the ſecond term in both equations, is n or nr^2, the ſum of the roots.

Cor. 2. *The product of the ſquares of the odd cords + the product of the ſquares of the even cords, that is,* $BD^2 \times BF^2 \times BH^2 + BE^2 \times BG^2 \times BI^2 = 4CB^n$.

Cor. 3. *Hence alſo if the circumference be divided into an even number of equal parts as n; and cords be drawn from any point* B *to all the reſt, then the ſum of the ſquares of the alternate cords will be equal to each other, and to* $n \times CB^2$; $BA^2 + BF^2 + BH^2 + BL^2 + BN^2 = BE^2 + BG^2 + BM^2 + BO^2 = n \times CB^2$. 16.

This follows from Cor. 1, ſuppoſing the point G to coincide with B, and D with A in Fig. 14, 15.

FIG. 16. Cor. 4. *If the arch* AR, *the multiple of* AD, *be greater than half the circumference; then the cord of the ſupplement will be* $-a$. *In which caſe you muſt write* $-$ BR *for* $+$ BR, *and* $+$ BR *for* $-$ BR, *in the preſent propoſition.*

SCHOLIUM.

Since the ſquare of the cord = verſed ſine × diameter, therefore, in the foregoing propoſitions, and their corollaries, you may, by ſubſtitution, find the ſums or products of the verſed ſines of arches, inſtead of the ſums or products of the ſquares of the cords correſponding to them.

PROP. XLIV.

17. 18. *If the circumference of a circle, whoſe diameter is* AR, *and center* O, *be divided into any number of equal parts as* n, *at the points* A, C, E, G, *&c, beginning at* A; *and from any point* P *in the diameter* AR, *be drawn the lines* PC, PE, PG, *&c: I ſay, the product of all the lines* AP × PC × PE × PG *&c thro' the whole circle is* = $AO^n - OP^n$.

For draw the cords AC, AE, AG; and from C, E, &c, let fall perpendiculars upon the diameter AR, as CI, &c, and let AO or OR $= r$, PO $= x$, AP $= y = r - x$. Then $PC^2 = AP^2 + AC^2 - 2PA \times$

$$\left.\begin{array}{l} AI = yy + AC^2 - \frac{y}{r}AC^2 \\ \text{that is, } PC^2 = y^2 + \frac{x}{r}AC^2 \\ \text{likewiſe } PE^2 = y^2 + \frac{x}{r}AE^2 \\ \text{alſo } PG^2 = y^2 + \frac{x}{r}AG^2, \text{ \&c} \end{array}\right\} \text{theſe multiplied}$$

together, produce $PC^2 \times PE^2 \times PG^2 = y^6 + \frac{x}{r}y^4 \times$

AC

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$\overline{AC^2 + AE^2 + AG^2} + \frac{xx}{rr}y^2 \times \overline{AC^2 \times AE^2 + AC^2 \times AG^2}$ FIG

$\overline{+ AE^2 \times AG^2} + \frac{x^3}{r^3} \times AC^2 \times AE^2 \times AG^2 = PC \times Pc \times$

$PE \times Pe \times PG \times Pg$. But in the ſecond term of this equation, $AC^2 + AE^2 + AG^2$ is the ſum of the ſquares of all the cords (AC, AE, AG) which call A. And in the third term is the ſum of all their rectangles $= B$. In the fourth term, the ſum of all the ſolids $= C$, and ſo on. Likewiſe the index of y in the firſt term is $n - 1$, when n is odd, or $n - 2$ when even. This being premiſed; the equation will be expreſſed in general thus, $PC \times Pc \times PE \times Pe$ &c $= y^{n-1} + \frac{x}{r}y^{n-3}A + \frac{xx}{rr}y^{n-5}B + \frac{x^3}{r^3}y^{n-7}C + \frac{x^4}{r^4}y^{n-9}D$ &c when n is odd. And $PC \times Pc \times PE \times Pe$ &c $= y^{n-2} + \frac{x}{r}y^{n-4}A + \frac{xx}{rr}y^{n-6}B + \frac{x^3}{r^3}y^{n-8}C$ &c when n is even. Let $x =$ cord of AC, AE, &c; then,

CASE I.

If n be an odd number; then by Cor. 1 Prop. XXIX, the cord of n times that arch $= \frac{x^n}{r^{n-1}} - \frac{nx^{n-2}}{r^{n-3}}$ &c $= 0$ in this caſe. But here one root is $x = 0$, by which dividing the equation, and multiplying by r^{n-1}, you will have $x^{n-1} - nr^2x^{n-3} + \frac{n.\overline{n-3}}{2}r^4x^{n-5} - \frac{n.\overline{n-4}.\overline{n-5}}{6}r^6x^{n-7}$ &c $= 0$, wherein the roots are AC, AE, AG, Ag, Ae, Ac. But ſince $AC = Ac$, $AE = Ae$, $AG = Ag$, and ſince in this equation there are only the even powers of x; therefore xx is the al- 17.

FIG. 17. gebraic root of it, and all the particular roots are AC², AE², AG².

But in every equation, the coefficient of the ſecond term is = the ſum of all the roots, therefore AC² + AE² + AG² = nr^2 = A. The coefficient of the third term = ſum of all the rectangles = $\frac{n.\overline{n-3}}{2}r^4$ = B, the coefficient of the fourth term = ſum of all the ſolids = C = $\frac{n.\overline{n-4}.\overline{n-5}}{6}r^6$, and ſo on, Therefore putting for A, B, C, &c, theſe their values, and we ſhall have PC × PE × $\overline{\text{PG} \times}$ Pg &c =

$$y^{n-1}+\frac{x}{r}y^{n-3}\times nr^2+\frac{xx}{rr}y^{n-5}\times\frac{n.\overline{n-3}}{2}r^4 \text{ \&c} =$$

$$y^{n-1}+nrxy^{n-3}+\frac{n.\overline{n-3}}{2}r^2x^2y^{n-5}+$$

$$\frac{n.\overline{n-4}.\overline{n-5}}{6}r^3x^3y^{n-7}$$ &c, that is (putting $r-x$ for y and involving)

$$=r^{n-1}-\overline{n-1}.r^{n-2}x+\frac{\overline{n-1}.\overline{n-2}}{2}r^{n-3}x^2-\frac{\overline{n-1}.\overline{n-2}.\overline{n-3}}{6}r^{n-4}x^3 \text{ \&c}$$

$$+nr^{n-2}x-\frac{n.\overline{n-3}}{1}r^{n-3}x^2+\frac{n.\overline{n-3}.\overline{n-4}}{2}r^{n-4}x^3 \text{ \&c}$$

$$+\frac{n.\overline{n-3}}{2}r^{n-3}x^2+\frac{n.\overline{n-3}.\overline{n-5}}{2}r^{n-4}x^3 \text{ \&c}$$

$$+\frac{n.\overline{n-4}.\overline{n-5}}{6}r^{n-4}x^3 \text{ \&c}$$

$$+\ldots\ldots x^{n-1}$$

$$=r^{n-1}+r^{n-2}x+r^{n-3}x^2+r^{n-4}x^3 \text{ \&c.}$$

$$\ldots+x^{n-1}$$

Then multiply by PA or $r-x$, and you will have PA × PC × PE × PG × Pg × Pe × Pc = r^n-x^n.

CASE

CASE II.

If n be an even number; then by Cor. I Prop. xxix, the cord of n times the arch $= \sqrt{4rr - xx} \times \frac{x^{n-1}}{r^{n-1}} - \frac{\overline{n-2}\,.\,x^{n-3}}{r^{n-3}}$ &c $= 0$, in which is one root $x = 0$, and dividing the equation by x and the correſpondent cord of its ſupplement $\sqrt{4rr - xx}$ (which here is AR), and multiplying by r^{n-1}, you will have $x^{n-2} - \frac{\overline{n-2}}{1} r^2 x^{n-4} + \frac{\overline{n-3}\,.\,\overline{n-4}}{2} r^4 x^{n-6} - \frac{\overline{n-4}\,.\,\overline{n-5}\,.\,\overline{n-6}}{6} r^6 x^{n-8}$ &c $= 0$, in which there being only the even powers of x, the roots are AC^2, AE^2, AG^2. FIG. 18.

But in this equation, the coefficient of the ſecond term is = ſum of all the roots $AC^2 + AE^2 + AG^2 = \overline{n-2} \times r^2 = A$. The coefficient of the 3d term is the ſum of all their rectangles $= \frac{\overline{n-3}\,.\,\overline{n-4}}{2} r^4 = B$. The coefficient of the fourth term is the ſum of all the ſolids $= \frac{\overline{n-4}\,.\,\overline{n-5}\,.\,\overline{n-6}}{6} r^6 = C$, and ſo on. Therefore putting for A, B, C, &c, their values in the general equation, we have $PC \times PE \times PG \times Pg$ &c $= y^{n-2} + \frac{x}{r} y^{n-4} \times \overline{n-2}\,. r^2 + \frac{x^2}{r} y^{n-6} \times \frac{\overline{n-3}\,.\,\overline{n-4}}{2} r^4$ &c $= y^{n-2} + \overline{n-2}\,. r x y^{n-4} + \frac{\overline{n-3}\,.\,\overline{n-4}}{2} r^2 x^2 y^{n-6} + \frac{\overline{n-4}\,.\,\overline{n-5}\,.\,\overline{n-6}}{6} r^3 x^3 y^{n-8}$ &c, that is (putting $r - x$ for y and involving)

$$\begin{array}{llll}
= r^{n-2} - \overline{n-2}\,.\,r^{n-3}x & + \dfrac{\overline{n-2}\,.\,\overline{n-3}}{2} r^{n-4}x^2 & - \dfrac{\overline{n-2}\,.\,\overline{n-3}\,.\,\overline{n-4}}{6} r^{n-5}x^3 \text{ \&c} \\
\quad + \overline{n-2} & - \dfrac{\overline{n-2}\,.\,\overline{n-4}}{1} & + \dfrac{\overline{n-2}\,.\,\overline{n-4}\,.\,\overline{n-5}}{2} \text{ \&c} \\
 & + \dfrac{\overline{n-3}\,.\,\overline{n-4}}{2} & - \dfrac{\overline{n-3}\,.\,\overline{n-4}\,.\,\overline{n-6}}{2} \text{ \&c} \\
 & & + \dfrac{\overline{n-4}\,.\,\overline{n-5}\,.\,\overline{n-6}}{6} \text{ \&c} \\
 & & \qquad \text{\&c}
\end{array}$$

$$\ldots\ldots \text{\&c} + x^{n-2}$$

$$= r^{n-2} \quad * \quad + r^{n-4}x^2 \quad * \quad + \quad r^{n-6}x^4 \text{ \&c} \ldots \ldots + x^{n-2}$$

Multiply by $PA \times PR$ or $rr - xx$, and you will have, $PA \times PC \times PE \times PG \times PR \times Pg \times Pe \times Pc = r^n - x^n$. *Q. E. D.*

FIG. 18. Cor. 1. *If the ſemicircumference* AER *be divided into any number of equal parts as n; at the points* A, B, C, D, *&c, and from any point* P *in the diameter* AR *be drawn lines to all the even points of diviſion, as* PC, PE, PG; *I ſay, the product of* PA *into that of all the ſquares of theſe lines,* $PA \times PC^2 \times PE^2 \times PG^2$, *if n is an odd number, or of* PA *and* PR *into theſe ſquares,* $PA \times PR \times PC^2 \times PE^2 \times PG^2$ *if n is even, is* $= AO^n - OP^n$.

For if $n \times AC =$ whole circumference, then $n \times AB =$ half the circumference, and $Pc = PC$, $Pe = PE$ &c.

Cor. 2. *If a ſemicircle be divided into n equal parts, and one of them be* K, *and s, t, u, &c, be put for the coſines of* 2K, 4K, 6K, *&c, to* $n - 1\,.\,K$, *if n is odd, or to* $n - 2\,.\,K$ *if even; then I ſay,* $r - x \times rr - 2sx + xx \times rr - 2tx + xx \times rr - 2ux + xx$ *&c, if n be an odd number, or* $r - x \times r + x \times rr - 2sx + xx \times rr - 2tx + xx \times rr - 2ux + xx$ *&c, (if n be even)* $= r^n - x^n$.

For let a, b, c, &c, be the ſines of 2K, 4K, 6K, &c, or of AC, AE, AG, &c; then $CI = a$, $OI = s$, PI

=

$= s - x$, AP $= r - x$, PR $= r + x$, and PC$^2 = aa + \overline{s-x}^2 = aa + ss - 2sx + xx = rr - 2sx + xx$; after the ſame manner PE$^2 = rr - 2tx + xx$, and PG$^2 = rr - 2ux + xx$, &c; and by Cor. 1, the product of theſe into $r - x$, or elſe into $\overline{r-x} \times \overline{r+x}$, is $= r^n - x^n$. But obſerve that the coſines of arches greater than 90°, will be negative. FIG. 18.

Cor. 3. *Hence alſo if d, e, f, &c, be the ſines of* IK, 2K, 3K, *&c; then* $y \times \overline{y^2 + \frac{4ddx}{r}} \times \overline{y^2 + \frac{4eex}{r}} \times \overline{y^2 + \frac{4ffx}{r}}$ *&c, if n be odd,*

Or $y \times \overline{r+x} \times \overline{y^2 + \frac{4ddx}{r}} \times \overline{y^2 + \frac{4eex}{r}} \times \overline{y^2 + \frac{4ffx}{r}}$ *&c, if n be even,* $= r^n - x^n$.

This follows from the demonſtration of this Prop. for $2d =$ AC, $2e =$ AE, $2f =$ AG, &c.

Cor. 4. *Alſo, if a, b, c, &c, be the verſed ſines of* 2K, 4K, 6K, *&c;*

$$\left.\begin{array}{l}\text{then if } n \text{ be odd, } y\\ \text{if } n \text{ be even, } y \times \overline{r+x}\end{array}\right\} \times \overline{y^2 + 2ax} \times \overline{y^2 + 2bx} \times \overline{y^2 + 2cx}$$

&c $= r^n - x^n$. *For* $a = \frac{2dd}{r}$, $b = \frac{2ee}{r}$, $c = \frac{2ff}{r}$.

Cor. 5. *And if* P *be taken without the circle at* π, *the product of all the lines* πA × πC × πE × πG *&c* $=$ πO^n — AO^n.

For let Oπ : OA : : OA : OP. And let the lines PC, PE, &c, be drawn; then ſince πO : CO : : CO : OP, and angle O common, therefore the triangles πOC and COP are ſimilar, and for the ſame reaſon, πEO, and EOP, &c. Whence πO : CO : : πC : PC : : πE : PE, &c, : : πR : PR : : πA : PA. Therefore PA $= \frac{r \times \pi\text{A}}{\pi\text{O}}$, PR $= \frac{r \times \pi\text{R}}{\pi\text{O}}$, PC $= \frac{r \times \pi\text{C}}{\pi\text{O}}$, PE $= \frac{r \times \pi\text{E}}{\pi\text{O}}$, &c, PO $= \frac{rr}{\pi\text{O}}$. Whence AP × PC

×

FIG. $\times PE \times PG \ \&c = \frac{\pi A \times \pi C \times \pi E \times \pi G \ \&c}{\pi O^n} r^n = r^n - \frac{r^{2n}}{\pi O^n}$, or $\pi A \times \pi C \times \pi E \ \&c = \pi O^n - r^n$.

PROP. XLV.

17. 18. *If the circumference of a circle* AEG, *whoſe diameter is* AR, *and center* O, *be divided into any number of equal parts as* n, *at the points* B, D, F, &c, *and the part* B*b be biſected by the diameter* AR; *and from any point* P *therein there be drawn the lines* PB, PD, PF *&c. I ſay, the product of all theſe lines* $PB \times PD \times PF$ *&c, thro' the whole circle, is* $= AO^n + OP^n$.

For let all the parts BD, DF, &c, be biſected in C, E, &c, and PC, PE, &c, drawn, and then the circle will be divided into $2n$ parts, therefore

by the laſt prop. } $PA \times PB \times PC \times PD \times PE \ \&c = AO^{2n} - OP^{2n}$,
and by the ſame } $PA \times PC \times PE \ \&c = AO^n - OP^n$,
therefore by diviſion } $PB \times PD \times PF \times PR \times Pf \times Pd \times Pb$, when n is odd,
or $PB \times PD \times PF \times PH \times Pb \times Pf \times Pd \times Pb$,
when n is even, is $= \frac{AO^{2n} - OP^{2n}}{AO^n - OP^n} = AO^n + OP^n$.

Cor. I. *If the ſemicircumference* AER *be divided into any number of equal parts as* n, *in the points* A, B, C, D, *&c, and from any point* P *in the diameter* AR, *the lines* PB, PD, PF, *&c, be drawn to all the odd points of diviſion: then, the product of all their ſquares, if* n *is even,* $PB^2 \times PD^2 \times PF^2 \times PH^2$; *or that product* $\times d$ *by* PR, *that is,* $PB^2 \times PD^2 \times PF^2 \times PR$, *if* n *is odd, is* $= AO^n + OP^n$.

For if $n \times Bb =$ whole circumference, then $n \times AB = \frac{1}{2}$ circumference; and $Pb = PB$, $Pd = PD$, &c.

Cor. 2. *If a ſemicircle be divided into any number of equal parts, as* n, *and one of them be* K, *and*

s, t, u, &c, be the coſines of 1K, 3K, 5K, *&c, to* $\overline{n-1}$.K *when n is even, or to* $\overline{n-2}$.K *when n is odd.* FIG. 17. 18.
Then I ſay

$$\overline{rr-2sx+xx}\times\overline{rr-2tx+xx}\times\overline{rr-2ux+xx}\ \&c=r^n+x^n,$$

when n is even, or

$$\overline{r+x}\times\overline{rr-2sx+xx}\times\overline{rr-2tx+xx}\times\overline{rr-2ux+xx}\ \&c$$

$= r^n + x^n$, when n is odd.

For if a, b, c, &c, be the ſines of 1K, 3K, 5K, &c; then the ſame way as in Cor. 2 Pr. XLIV, will be found $PB^2=rr-2sx+xx$, $PD^2=rr-2tx+xx$, $PF^2=rr-2ux+xx$, &c; whoſe product alone, or elſe multiplied by $r+x$, is, by Cor. 1, $=r^n+x^n$.

If P *be without the circle as at* π, *it will ſtill be* $\pi B\times\pi D\times\pi F$ &c $= AO^n + O\pi^n$.

For by this Prop. and Cor. 5 of the laſt, that product will be $\dfrac{O\pi^{2n}-AO^{2n}}{O\pi^n-AO^n}=O\pi^n+AO^n$.

PROP. XLVI.

If the arch AE *of a circle be divided into any number of equal parts as n, of which* AB *is one; and if the whole circumference be divided into the ſame number of equal parts, at the points* B, C, D, E, *&c, beginning at* B; *and if from any point* P, *taken in the diameter* AR, *the lines* PB, PC, PD, &C, *be drawn, and* EF *be let fall perpendicular to* AR *to interſect it in* F, *between the center* O *and* R. *And if* $AO=r$, $OP=x$, $OF=b$; *then I ſay the product of all the ſquares* $PB^2\times PC^2\times PD^2\times PE^2$ &C $= r^{2n}+2br^{n-1}x^n+x^{2n}$. 20.

For let $AO=r=1$. $AP=y=1-x$, then you will find as in Prop. XLIV, $PB^2=y^2+x\times AB^2$, $PC^2=y^2+x\times AC^2$, $PD^2=y^2+x\times AD^2$, &c; which multiplied together, produce $PB^2\times PC^2\times PD^2$ &c $=y^v+Axy^{v-2}+Bx^2y^{v-4}+Cx^3y^{v-6}$ &c $+Fx^n$, putting $v=2n$, A = ſum of the ſquares of the cords

AB²

FIG. 20. $AB^2 + AC^2 + AD^2$ &c, $B =$ ſum of their rectangles, $C =$ ſum of their ſolids, &c, and F product of them all.

Now if z be the cord of an arch, c the cord of n times the arch, then by Prop. XXIX, $z^{\nu} - \nu z^{\nu-2} + \nu \cdot \frac{\nu-3}{2} z^{\nu-4} - \nu \cdot \frac{\overline{\nu-4} \cdot \overline{\nu-5}}{2 \cdot 3} z^{\nu-6}$ &c $= \pm cc$.

In which equation, the root zz repreſents indifferently the ſquare of any cord AB, AC, AD, &c. Therefore by common algebra, the ſum of the ſquares of the cords, $AB^2 + AC^2 + AD^2$ &c $= \nu$, the ſum of their products $= \nu \cdot \frac{\nu-3}{2}$; ſum of their ſolids $= \nu \times \frac{\overline{\nu-4} \cdot \overline{\nu-5}}{2 \cdot 3}$, &c, and the product of all $= cc$. Whence we get $A = \nu$, $B = \frac{\nu \cdot \overline{\nu-3}}{2}$, $C = \frac{\nu \cdot \overline{\nu-4} \cdot \overline{\nu-5}}{2 \cdot 3}$, &c, and $F = cc$. Conſequently $PB^2 \times PC^2 \times PD^2$ &c $= y^{\nu} + \nu x y^{\nu-2} + \nu \cdot \frac{\nu-3}{2} x^2 y^{\nu-4} + \frac{\nu \cdot \overline{\nu-4} \cdot \overline{\nu-5}}{2 \cdot 3} \times x^3 y^{\nu-6}$ &c $+ ccx^n$; and putting $1 - x$ for y, and involving, we have $PB^2 \times PC^2 \times PD^2$ &c $=$

$$\begin{array}{l}
1 - \nu x + \nu \cdot \frac{\nu-1}{2} x^2 - \nu \cdot \frac{\nu-1}{2} \cdot \frac{\nu-2}{3} x^3 \text{ \&c} \ldots \text{\&c} - \nu x^{\nu-1} + x^{\nu} \\
+ \nu x - \nu \cdot \frac{\nu-2}{1} x^2 + \nu \cdot \frac{\nu-2}{1} \cdot \frac{\nu-3}{2} x^3 \text{ \&c} \qquad \text{\&c} + \nu x^{\nu-1} \\
\qquad + \nu \cdot \frac{\nu-3}{2} x^2 - \nu \cdot \frac{\nu-3}{2} \cdot \frac{\nu-4}{1} x^3 \text{ \&c} \\
\qquad\qquad\qquad\qquad \text{\&c} \\
\qquad\qquad\qquad\qquad + ccx^n.
\end{array}$$

Hence it will appear, by ſumming them up, that the coefficients of the ſecond, third, &c, terms are 0, till you come to the middle term, where you have

ccx^n (the index n being $\frac{1}{2}\nu$); likewiſe beyond the middle term they are o, becauſe the coefficients are the ſame at equal diſtances on each ſide from the middle; whence they all vaniſh till the laſt x^ν. FIG. 20.

Now to find the coefficient of the middle term, it muſt be obſerved, that ccx^n ſupplies the place of the firſt term of the next horizontal ſeries; which term, if it were there, then the ſum of the coefficients (in that perpendicular column) would there alſo be $= 0$, as before. Hence it follows, that if $\nu = 2, 4, 6,$ &c, that the term νx, $\nu . \frac{\nu - 3}{2} x^2$, $\nu . \frac{\nu - 4}{2} . \frac{\nu - 5}{3} x^3$, &c, will be wanting, reſpectively, which term is $2x$, $2x^2$, $2x^3$, &c, reſpectively. And in general, the term wanting is always $\frac{1.2.3 \ldots \overline{n-1} \times \nu}{1.2.3 \ldots \overline{n-1} . n} x^n = \frac{\nu}{n} x^n = 2x^n$. Conſequently the middle term will be found to be $- 2x^n + ccx^n$. But $cc = \text{AR} \times \text{AF} = 2 + 2b$, therefore $- 2x^n + \overline{2 + 2b} . x^n = 2bx^n$ is the middle term. And therefore $\text{PB}^2 \times \text{PC}^2 \times \text{PD}^2$ &c, thro' the whole circle, $= 1 + 2bx^n + x^{2n}$, or rather $r^{2n} + 2r^{n-1}bx^n + x^{2n}$.

Cor. 1. *If the arch* AE *be leſs than a quadrant, and* F *lie from* O *towards* A; *then will* $\text{PB}^2 \times \text{PC}^2 \times \text{PD}^2 \times \text{PE}^2$ &c $= r^{2n} - 2br^{n-1}x^n + x^{2n}$. 21.

This follows from this Prop. putting $-b$ inſtead of $+b$, as lying on the contrary ſide of o.

Cor. 2. *If* P *be without the circle as at* π, *then will* $\pi\text{B}^2 \times \pi\text{C}^2 \times \pi\text{D}^2$ &c $= r^{2n} \pm 2br^{n-1}x^n + x^{2n}$, *according as* F *falls beyond, or on this ſide of* O, *in reſpect to* P. 20. 21.

The demonſtration of this Prop. equally proves this, ſince the even powers of $x - 1$, are the ſame with thoſe of $1 - x$, and ν or $2n$ is an even number.

Cor.

FIG. 21. Cor. 3. *Let* b *= coſine of an arch* A *or* AE, *to the radius* r; *and* $\frac{A}{n} = B$, $\frac{360}{n} = C$, *and* s, t, u, &c, *the coſines of* B, B + C, B + 2C, B + 3C, &c, to $B + \overline{n - 1}.C$, *to the radius* r, $x =$ *any quantity; then I ſay,* $rr - 2sx + xx \times rr - 2tx + xx \times rr - 2ux + xx$ &c, *to* n *terms, is* $= r^{2n} - 2br^{n-1}x^{n} + x^{2n}$.

For let $a =$ BI the ſine of B, $s =$ OI the coſine, then $PB^2 = BI^2 + PI^2 = aa + \overline{s - x}^2 = aa + ss - 2sx + xx = rr - 2sx + xx$, for the ſame reaſon $PC^2 = rr - 2tx + xx$, $PD^2 = rr - 2ux + xx$, &c; but by Cor. 1, the product of theſe is $= r^{2n} - 2br^{n-1}x^{n} + x^{2n}$.

If A be greater than a quadrant, write $+ b$ for $- b$. Likewiſe any of the coſines, s, t, u, &c, of arches between 90 and 270, is negative, and therefore in that caſe write $+ s$, $+ t$, $+ u$ inſtead of $- s$, $- t$, $- u$, &c.

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THE ELEMENTS OF TRIGONOMETRY.

BOOK II.

Plain trigonometry, or the doctrine of plain triangles.

PLAIN trigonometry teaches the relation of the ſides and angles of plain triangles, and how to calculate their ſides and angles.

A plain triangle is made by three right lines, which are called it's ſides, and if one ſide be perpendicular to another, or makes a right angle with it; then it is called *a right-angled triangle,* otherwiſe it is an oblique triangle.

A right angle is meaſured by an arch of 90°, an *acute angle* is leſs, and an *obtuſe* one greater than a right angle.

In a right-angled triangle, the ſide oppoſite to the right angle is called the *hypothenuſe*; and the other two, the *legs* or *ſides,* or ſometimes the *perpendicular and baſe.*

In all plain triangles, the ſum of the three angles together is 180 degrees: and in a right-angled triangle, the ſum of the two oblique angles is 90°.

When three letters denote an angle, the middle letter is always at the angle.

SECT.

SECT. I.

The relations, and ſeveral proportions among the ſides, and angles of plain triangles.

PROP. I.

FIG. 1. *In any right-angled triangle* CAB, *let the hypotheneuſe* CA $= b$, *and* CA $-$ CB *or* BD $= v$; *then,*

$$57.29578\sqrt{\frac{2v}{b}} + \frac{v}{3.4b}\text{A} + \frac{3^2v}{5.8b}\text{B} + \frac{5^2v}{7.12b}\text{C} \;\&\text{c}$$

$=$ *angle* ACB *in degrees; where* A, B, C *are the preceding terms.*

For with the center C deſcribe the arch AD, then BD or v is the verſed ſine of the arch AD, and radius b; therefore (by Prop. XIV, B. I,) arch AD $=$ $\sqrt{2bv} + \frac{v}{3.4b}\text{A} + \frac{3^2v}{5.8b}\text{B} + \frac{5^2v}{7.12b}\text{C}$ &c; but as half the circumference $3.14159b$: 180 degrees :: AD : $\frac{180\,\text{AD}}{3.14159b}$ $=$ degrees in the arch AD $= \frac{57.295779}{b}$ $\times$ AD, that is, $\angle$ ACD $= \frac{57.29}{b} \times \sqrt{2bv} + \frac{v}{3.4b}\text{A}$ &c $= 57.2957\sqrt{\frac{2v}{b}} + \frac{v}{3.4b}\text{A} + \frac{3^2v}{5.8b}\text{B}$ &c.

Cor. 1. *As the ſum of the hyp. and* ½ *longer leg*
AC $+$ ½ CB :
To 86 ::
So the ſhorter leg AB :
To its oppoſite angle C, *nearly.*

Let CB $= b$, and AB $= c$, $a = \angle$ C in degrees, then by the nature of the circle, $c = \sqrt{2bv - vv} =$

$\sqrt{2bv}$

$\surd 2bv - \frac{v}{4b}\surd 2bv$ &c, by evolution. Then by this prop. $a = \frac{57.3}{b} \times : \sqrt{2bv} + \frac{v}{12b}\sqrt{2bv} :$ nearly; multiply this equation by the former, and you have $a \times : \sqrt{2bv} - \frac{v}{4b}\surd 2bv = \frac{57.3c}{b} \times : \surd 2bv + \frac{v}{12b}\sqrt{2bv}$, and dividing by $\surd 2bv$, $a \times \overline{1 - \frac{v}{4b}} = \frac{57.3c}{b} \times \overline{1 + \frac{v}{12b}}$.

FIG. 1.

And $\frac{57.3c}{b} = a \times \frac{1 - \frac{v}{4b}}{1 + \frac{v}{12b}} =$ (by diviſion) $a \times \overline{1 - \frac{v}{3b}}$ &c $= a \times \frac{3b - v}{3b} = a \times \frac{3b - b + b}{3b} = a \times \frac{2b + b}{3b}$, whence $a \times \overline{2b + b} = 171.9c = 172c$ nearly, or $\overline{b + \frac{1}{2}b} \times a = 86c$.

SCHOLIUM.

If $c = 1$, $n = 57.3$, then $\frac{3}{2}n = 86$, and by Cor. 1, $b + \frac{1}{2}b : 1 :: \frac{3}{2}n : a$, and $2b + b = \frac{3n}{a}$, and $b = \frac{3n}{2a} - \frac{b}{2} = \frac{3n}{2a} - \frac{\sqrt{bb - 1}}{2}$, whence $\frac{3n}{a} - 2b = \sqrt{bb - 1}$. This equation reduced gives $b = \frac{2n}{a} - \sqrt{\frac{nn}{aa} - \frac{1}{3}}$. Whence $b = \frac{3n}{a} - 2b$; which is *Forſter*'s rule.

PROP. II.

FIG. 1. *In any right-angled triangle* CAB, *let* $a =$ *degrees in the leſſer* ∠ C, $m = \frac{a}{57.29578}$. *Then*

As 1 :

To $1 - \frac{m^2}{1.2}\text{A} - \frac{m^2}{3.4}\text{B} - \frac{m^2}{5.6}\text{C} - \frac{m^2}{7.8}\text{D}$ &c ::

So the hypothenuſe CA :

To the adjoining ſide CB.

where A, B, C, &c, are the foregoing terms, with their ſigns.

For let radius CA or CD $= b$. Arch AD $= z$, then $z = \frac{ab}{57.295\ \&c} = mb$, and by Cor. 1 Pr. XII B. I, coſine CB $= b - \frac{z^2}{1.2b} + \frac{z^4}{1.2.3.4b^3}$ &c $= b - \frac{m^2b^2}{1.2b} + \frac{m^4b^4}{1.2.3.4b^3} - \&c = \text{CA} \times 1 - \frac{m^2}{1.2} + \frac{m^4}{1.2.3.4} - \frac{m^6}{1\ldots6} + \frac{m^8}{1\ldots.8}$ &c.

Cor. 1. *If* a *be the leſſer angle, in degrees, then it will be,*

As $\frac{57.3}{a} + \frac{3a}{1000} : 1 ::$ *hyp.* CA : *op. ſide* AB, *nearly.*

For let $n = 57.3$, then $\text{CB}^2 = bb - z^2 + \frac{z^4}{3b^2}$ &c, and $\text{AB}^2 = \text{CA}^2 - \text{CB}^2 = z^2 - \frac{z^4}{3b^2}$, and extracting the root, $\text{AB} = z - \frac{z^3}{6bb} = mb - \frac{m^3b}{6} = b \times$

$\frac{a}{n}$

$\overline{\frac{a}{n}-\frac{a^3}{6n^3}}$; and dividing both ſides by $\frac{a}{n}-\frac{a^3}{6n^3}$, $b =$ FIG. 1.

$\text{AB} \times \overline{\frac{n}{a}+\frac{a}{6n}} = : \frac{57.3}{a} + \frac{3a}{1031} \times : \text{AB}.$

Cor. 2. *If a be the leſſer angle in degrees, then As* 1 : $1-\frac{1\frac{1}{2}aa}{10000}$ :: *hyp.* CA : *adjoining ſide* BC, *nearly.*

For $\text{CB} = \text{CA} \times \overline{1-\frac{m^2}{2}} = \text{CA} \times \overline{1-\frac{aa}{2nn}} = \text{CA} \times \overline{1-\frac{1.52aa}{10000}}$.

SCHOLIUM.

Let $q = 90°$, $y = q - a$, then ſince $\text{CB} = \text{CA} \times \overline{1-\frac{aa}{2nn}}$, by diviſion $\text{CA} \doteq \text{CB} \times \overline{1+\frac{aa}{2nn}}$, and $\text{CA} \times y = \text{CB} \times \overline{y+\frac{4aa}{\frac{8nn}{y}}}$. But $\frac{8nn}{y} \doteq \frac{8nn}{q-a} = \frac{8nn}{q} + \frac{8nna}{qq}$ &c $= 292 + 3a$. Therefore $\text{CA} \times y = \text{CB} \times \overline{y+\frac{4aa}{300+3a}}$, nearly, which is *Wilſon's* rule.

Since in a right-angled triangle, the ſquare of the hypothenuſe is equal to the ſum of the ſquares of the ſides. And in any triangle, the baſe is to the ſum of the ſides, as their difference, to the difference of the ſegments made by a perpendicular. Therefore by theſe and the two foregoing propoſitions and their corollaries, all the caſes of plain triangles may be reſolved, without any tables whatever: dividing oblique triangles into two right-angled ones, by a

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FIG. perpendicular let fall from the end of a given ſide, and oppoſite to a given angle.

PROP. III.

1. *In any right-angled triangle* CAB, *any one of the three ſides being made a radius; and a circle deſcribed from the end of it as a center; then the other ſides will repreſent ſines, tangents, ſecants, &c, of the reſpective angles: then it will be, as any ſide : to what it repreſents in the table :: ſo is any other ſide : to what it repreſents in the table.*

Deſcribe the arches AD and BE, from the center C; then if CA be made radius, AB will be the ſine, and CB the coſine, of the angle C. And if CB be made radius, AB will be the tangent, and CA the ſecant of the angle C. And becauſe the ſines, tangents, &c, of any angles may be found in the tables; therefore, when any triangle is propoſed, there may always be found a ſimilar one in the tables; and conſequently the ſides about the equal angles are proportional. Therefore it will be,

As radius : AC :: ſin. C : AB :: coſ. C : CB.
and rad. : CB :: tan. C : AB :: ſec. C : CA.

Cor. 1. *If two right-angled plane triangles, have the ſame or equal hypothenuſes; the perpendiculars are as the ſines of the angles at the baſe; and the baſes are as the ſines of the angles at the vertex.*

For they are both in the given ratio of hypothenuſe to the radius.

Cor. 2. *If two right-angled plane triangles have equal baſes; the perpendiculars are as the tangents of the angles at the baſe; and the hypothenuſes are reciprocally as the ſines of the angles at the vertex.*

For the perpendicular and tan. oppoſite angles, are in the given ratio of the baſe to the radius.

And the hypothenuſes are as the ſecants of the angles at the baſe, that is reciprocally as the ſines of the angles at the vertex.

Cor. 3. *If two right-angled plane triangles have equal perpendiculars; the baſes are as the tangents of the angles at the vertex; and the hypothenuſes are reciprocally as the ſines of the angles at the baſe.*

This is plane by Cor. 2, making the baſe to become the perpendicular.

PROP. IV.

In any right-angled triangle, CDE, 2.
As the ſum of the hyp. and one ſide, CE + ED :
Radius : :
So the other ſide, CD :
To tan. half the op. angle, CED.

Produce DE to B, ſo that EB = EC, then ∠ B = ∠ BCE, and angle DEC = ∠ B + BCE = 2 ∠ B. But in the triangle CDB, by Prop. III, it is, as DB : radius : : CD : tan. B; that is, DE + EC : rad. : : CD : tan. $\frac{1}{2}$E.

PROP. V.

The ſides of any triangle, are proportional to the ſines of their oppoſite angles, AB : S. F : : AF : S. B : : FB : S. A. 3.

On AB, AF let fall the perpendiculars FH, B*b*; then by Prop. III, AF : FH : : R : S. A
FH : FB : : S. B : R
therefore ex equo, AF : FB : : S. B : S. A.

Again,

As AB : B*b* : : R : S. A
and B*b* : BF : : S. F : R
and ex equo, AB : BF : : S. F : S. A.

Cor. 1. *If a line* AD *be drawn from the vertex of any triangle cutting the baſe; the rectangle of the ſides* 4.

FIG. *and ſines of the vertical angles, are directly as the ſegments of the baſe.*

For $\text{S.CDA} : \text{CA} :: \text{S.CAD} : \text{CD} = \frac{\text{CA} \times \text{S.CAD}}{\text{S.CDA}}$.

And S.CDA or $\text{BDA} : \text{AB} :: \text{S.DAB} : \text{DB} = \frac{\text{AB} \times \text{S.DAB}}{\text{S.CDA}}$, therefore $\text{CD} : \text{DB} :: \text{CA} \times \text{S.CAD} : \text{AB} \times \text{S.DAB}$.

5. Cor. 2. *If in a triangle two ſides* AC, CB, *and the angle* A *oppoſite to the leſſer ſide* CB, *be given; the angle oppoſite to the greater may be either leſs or greater than a right angle; alſo the baſe and its oppoſite angle, will have each of them, two ſeparate values. Therefore the three things required are ambiguous.*

For make $\text{CD} = \text{CB}$, then $\angle\,\text{CDB} = \angle\,\text{B}$, and there are the two triangles ACB, ACD, wherein two ſides and an oppoſite angle are given the ſame in both; therefore the other three things may be found in either triangle; for the angle oppoſite to AC may be either B or it's ſup. ADC, and the baſe may be either AB or AD, and it's op. angle may be either ACB or ACD.

PROP. VI.

6. *In any triangle* ABC,

As the ſum of any two ſides $\text{BC} + \text{AB}$:
To their difference $\text{BC} - \text{AB}$ ::
Tan. half the ſum of their op. angles, tan. $\frac{\text{A} + \text{C}}{2}$:
Tan. half their difference, tan. $\frac{\text{A} - \text{C}}{2}$.

Make $\text{BD} = \text{BA}$, and draw AD, and BFG $\perp$ to it; likewiſe $\text{FE} \parallel \text{AC}$, then will $\text{AF} = \text{FD}$, $\text{DE} = \text{EC}$. The $\angle\,\text{BAF} = \frac{\text{BAD} + \text{BDA}}{2} = \frac{\text{BAC} + \text{C}}{2}$, and $\angle\,\text{CAF} = \text{BAC} - \text{BAF} = \frac{\text{BAC} - \text{C}}{2}$.

The triangles BFE, BGC are ſimilar, whence BE : EC : : BF : FG. But AF : R : : BF : tan. BAF : : FG : tan. FAG, or BF : FG : : tan. BAF : tan. FAG. Conſequently BE or $\frac{\text{BD}+\text{BC}}{2}$: EC or $\frac{\text{BC}-\text{BD}}{2}$: : tan. BAF : tan. FAG.

Otherwiſe.

BC : BA : : S.A : S.C, and BC + BA : BC — BA : : FIG.
S.A + S.C : S.A — S.C : : (by Cor. I Pr. VII) tan. 6.
$\frac{\text{A}+\text{C}}{2}$: tan. $\frac{\text{A}-\text{C}}{2}$.

Note, Inſtead of the tan. half the ſum of the op. angles, you may take the cotangent of half the included angle, which is equal to it.

Cor. I. *As the baſe* CA :
Sum of the ſides, CB + BA : :
So diff. ſides CD :
Diff. ſegments of the baſe made by a perpendicular (or the alternate baſe) : :
So coſ. ½ ſum op. angles, or S. ½ the vertical angle :
Coſ. ½ diff. op. angles, or coſ. ½ diff. vertical angles, made by a perp.

For ſince DF = ½ DA, FE = ½ AC; and in the triangle FBE, FE : BE : : S.FBD or coſ. BDF : S.BFE or GFE or coſ. DFE. But DFE or DAC = BDA — C = BAD — C, and 2 DAC = BAC — C, alſo ½ B = comp. $\frac{\text{A}+\text{C}}{2}$, and A — C = diff. vert. ∠ s made by a perpendicular.

Likewiſe it is evident from geometry, that CA : CB + BA : : CD : diff. ſegments of the baſe. Suppoſing a circle deſcribed with center B and radius BA.

FIG. 6.

Cor. 2. *As the baſe* CA :
Difference of the ſides CD : :
So ſum of the ſides CB + BA :
Diff. ſegments by a perpendicular (or the alternate baſe) : :
So ſine half ſum op. angles, $\frac{A+C}{2}$:
Sine of half their difference, $\frac{A-C}{2}$.

For in the triangle ADC, AC : DC : : S.ADB : S.CAD.

Cor. 3. *As ſquare of the baſe* AC^2 :
Difference of the ſquares of the ſides, $BC^2 - BA^2$: :
S. ſum op. angles, A + C :
S. their difference, A — C.

For by multiplying the two proportions in Cor. 1 and 2, $AC^2 : \overline{CB + BA} \times \overline{CB - BA} :: \text{coſ.}\,\frac{A+C}{2} \times S.\,\frac{A+C}{2} : \text{coſ.}\,\frac{A-C}{2} \times S.\,\frac{A-C}{2} ::$ (by Cor. 3 Pr. II B. I) $S.\overline{A+C} : S.\overline{A-C}$.

Cor. 4. *As half the ſum of the ſides* :
Tan. half the ſum of the op. angles : :
Greater ſide :
Tan. ½ *ſum* + *tan.* ½ *diff. op. angles* : :
Leſſer ſide :
Tan. ½ *ſum* — *tan.* ½ *diff. op. angles.*

For ſince $BC + BA : BC - BA :: T.\frac{A+C}{2} : T.\frac{A-C}{2}$.

Therefore, by compoſition and diviſion, BC + BA :

$$\left\{\begin{matrix} 2BC :: \\ 2BA :: \end{matrix}\right\} T.\frac{A+C}{2} : \left\{\begin{matrix} T.\frac{A+C}{2} + T.\frac{A-C}{2} \\ T.\frac{A+C}{2} - T.\frac{A-C}{2}. \end{matrix}\right.$$

Cor.

FIG. 70.

Cor. 5. In any triangle ACB, if CD be drawn biſecting the vertical angle C; it will be,

As twice the rectangle of the ſides, 2ACB :
To the rectangle of the biſecting line and ſum of the ſides, CD × $\overline{AC + CB}$ ::
So radius :
Coſ. half the vertical angle C.

For produce AC, and make CF = CB, and draw FB and CP ⊥ to it. Then AF = AC + CB, and ∠ F = ∠ CBF, and ∠ ACB = F + CBF = 2F, or ACD = F. And the triangles ACD, AFB are ſimilar; therefore AC : CD :: AF : FB = $\frac{CD \times AF}{AC}$. And CF (CB) : rad. :: FP ($\frac{1}{2}$FB) : s.FCP, or coſ. F; that is,

$$CB : \frac{CD \times AF}{2AC} :: \text{rad.} : \text{coſ.}\ \tfrac{1}{2} ACB.$$

Cor. 6. *As the leſſer ſide : to the greater* ::
So radius : to tan. of an arch A.
From A take 45°, then
As radius : tan. $\overline{A - 45}$, *the remaining arch* ::
So tan. ½ *ſum of the op. angles* :
To tan ½ *their difference.*

For let L be the leſſer, G the greater ſide, then ſince $\frac{rG}{L}$ = tan. A, we ſhall have (by Cor. 2 Pr. IX L. I) the tan. $\overline{A - 45} = \frac{\frac{rG}{L} - r}{\frac{rG}{L} + r} \times r = \frac{G - L}{G + L} r$, and therefore r : tan. $\overline{A - 45}$ :: G + L : G − L :: tan. ½ ſum of the oppoſite angles : tan. half their difference (by this prop.)

The

FIG. *The ſame will hold good, if you make, as the greater : to the leſſer ſide : : rad. : tan.* A, *and then take rad. : tan.* 45 — A : : *tan.* $\frac{1}{2}$*ſum : tan.* $\frac{1}{2}$ *diff. op. angles.*

PROP. VII.

7. *In any triangle* AFB,
As twice the rectangle of the legs, $2AF \times FB$:
Sum of ſquares of the legs — ſquare baſe : :
Radius :
Coſ. vertical angle, F.

On FB let fall the perpendicular AC, then (by Geom. II 23) $AB^2 + 2BF \times CF = AF^2 + FB^2$, and $2BF \times FC = AF^2 + FB^2 - AB^2$. But by Prop. III, AF : FC : : rad. : coſ. F. And multiplying the two firſt terms by 2FB, then will $2AF \times FB : 2BF \times FC$ or $AF^2 + FB^2 - AB^2$: : rad. : coſ. F.

Cor. 1. *Hence theſe are reſpectively proportional,*
As twice the rectangle of the legs :
Rectangle of the baſe + diff. legs, and baſe — diff. legs :
Rectangle of the ſum of the three ſides, and ſum legs — baſe : :
So radius :
Verſed ſine of the included angle :
Verſed ſine of the ſum of the oppoſite angles.

For let $m = AF$, $n = BF$, $b = AB$, then by this prop.

r : coſ. F : : $2mn : mm + nn - bb$; and by compoſition,

$r : r \pm$ coſ. F : : $2mn : 2mn \pm mm \pm nn \mp bb$,

that is, by ſchol. Prop. I,

$$r : \left\{\begin{matrix}\text{verſ. ſup. F}\\ \text{verſ. F}\end{matrix}\right\} : : 2mn : \left\{\begin{matrix}\overline{m+n}^2 - bb = \overline{m+n+b} \times \overline{m+n-b}\\ bb - \overline{m-n}^2 = \overline{b+\overline{m-n}} \times \overline{b-\overline{m-n}}.\end{matrix}\right.$$

FIG. 7.

Cor. 2. *From half the ſum of the three ſides, ſubtract each ſide ſeverally, and note the three remainders. Then,*

As half the rectangle of the legs :
To ſquare root of the product of this half ſum and the three remainders : :
So radius :
Sine of the vertical angle.

For let $s = b + m + n$, $v =$ verſ. F, $\mathrm{v} =$ verſ. ſup. F. Then by Cor. I,

$$v = \frac{\overline{b+m-n}\times\overline{b+n-m}}{2mn}r = \frac{\overline{s-2n}\times\overline{s-2m}}{2mn}r,$$

and $$\mathrm{v} = \frac{\overline{b+m+n}\times\overline{m+n-b}}{2mn}r = \frac{s\times\overline{s-2b}}{2mn}r.$$

And by Schol. I Prop. I,

$$\sqrt{\mathrm{v}v} = \mathrm{s.F} = \frac{r}{2mn}\sqrt{s\times\overline{s-2b}\times\overline{s-2m}\times\overline{s-2n}}$$

$$= \frac{2r}{mn}\times\sqrt{\frac{s}{2}\times\frac{s-2b}{2}\times\frac{s-2m}{2}\times\frac{s-2n}{2}}.$$

Cor. 3. *As the rectangle of the legs :*
Radius ſquare : :
So the rectangle ½ ſum 3 ſides — one leg, and that ½ ſum — other leg :
Sine ſquare of half the vertical angle : :
And ſo the rectangle ½ ſum 3 ſides, and that ½ ſum — baſe :
Coſine ſquare of half the vertical angle.

For multiplying the three laſt terms in Cor. I by $\frac{1}{2}r$; then $2mn : \overline{b+m-n}\times\overline{b+n-m} : s\times\overline{s-b} :: \frac{1}{2}rr : \frac{1}{2}rv : \frac{1}{2}r\mathrm{v}$. But by Prop. II, $\frac{1}{2}rv =$ ſine ſquare of half F, and $\frac{1}{2}r\mathrm{v} =$ coſ. ſquare of ½ F. Therefore $2mn : \overline{s-2m}\times\overline{s-2n} : s\times\overline{s-b} :: \frac{1}{2}rr :$ ſine^2 :

FIG. 7. ſine^2 : coſine^2. or $mn : \frac{s - 2m}{2} \times \frac{s - 2n}{2} : \frac{s}{2} \times \frac{s - b}{2}$: : rr : ſine^2 : coſine^2, of $\frac{1}{2}$F. And hence alſo follows,

Cor. 4. *As four times the rectangle of the legs :*
To radius ſquare : :
So rectangle of the baſe + diff. legs, and baſe — diff. legs :
Sine ſquare of half the vertical angle : :
And ſo rectangle of ſum legs + baſe, and ſum legs — baſe : :
Coſine ſquare of half the vertical angle.

Cor. 5. *Rectangle of ſum of the three ſides, and ſum legs — baſe :*
Rectangle baſe + diff. legs, and baſe — diff. legs : :
Radius ſquare :
Tangent ſquare of half the angle at the vertex.

This follows from Cor. 4, becauſe coſine : ſine : : rad. : tangent, by Prop. 1 B. 1.

SCHOLIUM.

Inſtead of the *rectangle* of the legs, you may take a *quarter of the rectangle,* of the *ſum* + the *difference* of *the legs*, into the *ſum* — *diff. legs*; which is equal thereto.

Likewiſe inſtead of *half the vertical angle,* you may take the *complement of half the ſum* of *the oppoſite angles*, which is alſo equal to it.

PROP.

PROP. VIII.

In any triangle AFB, *I ſay,* FIG. 8.

$\text{AB} - \frac{\text{AF} \times \textit{coſ.}\,\text{A}}{\textit{rad.}} : \text{S.A} :: \text{AF} : \textit{tan. oppoſite} \angle\, \text{B}.$

For let fall the perpendicular FR on the baſe; then in the triangle AFR, rad. : AF :: S.A : FR $= \frac{\text{AF} \times \text{S.A}}{r}$ :: coſ. A : AR $= \frac{\text{AF} \times \textit{coſ.}\,\text{A}}{r}$. And BR $= \text{AB} - \frac{\text{AF} \times \textit{coſ.}\,\text{A}}{r}$, then BR : rad. :: FR or $\frac{\text{AF} \times \text{S.A}}{r}$: tan. B. Or BR : S.A :: AF : tan. B.

Cor. $\sqrt{: \text{AB}^2 + \text{AF}^2 - \frac{2\text{AB} \times \text{AF}}{\textit{rad.}} \times \text{coſ.A} :} = \text{FB}.$
For $\text{FB}^2 = \text{AB}^2 + \text{AF}^2 - 2\text{AB} \times \text{AR}$.

PROP. IX.

In the triangle AFB, *draw* FR *perpendicular to the baſe, and make* BR = RD; *then,* 8.

As the baſe, AB :
To the difference of the ſegments AD ::
So ſine of the vertical angle F :
To ſine of the diff. vert. angles, or S. diff. angles at the baſe B, A.

For AB : S.AFB :: FB or FD : S.A :: AD : S.AFD.
And ∠ B — ∠ A = ∠ FDB — ∠ A = ∠ AFD.

PROP.

PROP. X.

FIG. 9. *As coſ. diff. angles at the baſe — coſ. ſum :*
Sine of the vertical angle : :
So perpendicular :
Half the baſe.

For let ADB be the triangle, DP the perpendicular, C the center of the circumſcribing circle, draw ECG perpendicular, and DF parallel to AB, &c. the ∠ ECB (= 2EBA) is the ſum of the angles A, B, whoſe verſed ſine is EG, and ECD or 2EBD is the difference of the angles A, B, whoſe verſed ſine is EF; therefore the difference of theſe verſed ſines, or the difference of the coſines, is FG or DP. Alſo GCB or ADB is the vertical angle, whoſe ſine is GB. And therefore DP : GB : : diff. coſines : S. vertical angle.

Cor. *As ſine of the diff. angles at the baſe :*
Sine of the vertical angle : :
Diſtance of the perp. from middle of the baſe :
Half the baſe.

For FD is the ſine of the difference of the angles A, B; and FD or GP : GB : : S. diff. : S. vert. angle.

PROP. XI.

8. *As radius :*
To ſum of the cotan. half the angles at the baſe : :
So the perpendicular :
To ſum of the three ſides.

Put $FR = p$, radius $= r$, then by Prop. IV, FIG. 8.

$\tan.\ \frac{1}{2}A : r :: r : \cot.\ \frac{1}{2}A :: p : AF + AR,$

and $r : \cot.\ \frac{1}{2}B :: p : BF + BR;$

$$\text{whence } AF + BF + AB = \frac{\cot.\ \frac{1}{2}A + \cot.\ \frac{1}{2}B}{r}p.$$

PROP. XII.

In a right-angled triangle ACB, 10

As radius :

Sine of double one acute angle, A *or* B *::*

Square of the hypothenuſe :

Four times the area.

Draw $CD \perp AB$, and biſect the hyp. AB in E, and draw EC; then $AE = EC = EB$, becauſe ACB may be inſcribed in a ſemicircle; then by ſimilar triangles, $AB : BC :: AC : CD = \frac{AC \times BC}{AB}$. Alſo $\angle CED = \angle B + BCE = 2B$, and $CEB = 2A$. Therefore by Prop. III, as rad : CE or $\frac{1}{2}AB :: s.E : CD$, or rad $: s.E :: \frac{1}{2}AB : \frac{AC \times CB}{AB} :: \frac{1}{4}AB^2 : \frac{AC \times CB}{2}$ or the area of the triangle.

PROP. XIII.

As radius :

To ſine of any angle of a triangle, A *::* 3

So half the rect. of the including ſides, $\frac{AF \times AB}{2}$:

To the area of the triangle.

FIG. For the area of the triangle AFB is ½ AB × FH;
3. but rad : AF :: S.A : FH $= \frac{AF \times S.A}{rad.}$, therefore the area $= \frac{AF \times AB \times S.A}{2\ rad.}$.

PROP. XIV.

9. *As coſ. diff. — coſ. ſum of the angles at the baſe :*
Sine of the vertical angle ::
So the ſquare of the perpendicular :
To the area of the triangle ::
And ſo the area of the triangle :
To the ſquare of half the baſe.

For by Prop. X, coſ. $\overline{B - A}$ — coſ. $\overline{B + A}$: S.D :: DP : GB :: DP^2 : GB × DP the area :: GB × DP : GB^2.

PROP. XV.

3. *In any triangle,*
As radius :
Tan. half the vertical angle ::
Rect. of half the perimeter, and that half — baſe :
To the area of the triangle.

For, let $s =$ S.AFB; c, t, τ the coſine, tangent, and cotangent of half AFB, $m =$ AF, $n =$ FB, AB $= b$, $z = \frac{m + n + b}{2}$. Then by Prop. XIII, $\frac{mns}{2r} =$ area, but by ſch. Prop. II B. I, $s = \frac{2cc}{\tau} = \frac{2cct}{rr}$, therefore $\frac{mncct}{r^3} =$ area. And by Cor. 3 Prop. VII, $\frac{mncc}{rr} = z \times \overline{z - b}$, therefore $\frac{z \times \overline{z - b}}{r} t =$ area.

Cor. *As radius :* FIG. 3.
Cotan. half the vertical angle : :
So rect. baſe + diff. ſides, and baſe — diff. ſides :
Four times the area of the triangle.

For if $a =$ s. $\frac{1}{2}$ angle F, then by ſch. Pr. II B. I, $s = \frac{2aa}{t} = \frac{2aa\tau}{rr}$, and the area $= \frac{mns}{2r} = \frac{mna^2\tau}{r^3}$. But by Cor. 3 Pr. VII, $mna^2 = \overline{z-m} \times \overline{z-n} \times rr = \frac{b+n-m}{2} \times \frac{b+m-n}{2} r^2$, therefore the area $= \frac{b+n-m}{2} \times \frac{b+m-n}{2r} \tau$.

PROP. XVI.

If three lines be drawn from the three angles of a triangle to any point O, *the product of the ſines of all the alternate angles will be equal.* 14

Let the ſines of the angles be as in the figure, then in the triangle.

$$\left.\begin{array}{lllllllll} \text{POI}, & a & : & f & :: & \text{IO} & : & \text{PO} \\ \text{IOH}, & e & : & d & :: & \text{HO} & : & \text{IO} \\ \text{HOP}, & c & : & b & :: & \text{PO} & : & \text{HO} \end{array}\right\} \text{theſe multiplied,}$$

$$aec : fdb :: \text{IO} \times \text{HO} \times \text{PO} : \text{PO} \times \text{IO} \times \text{HO}.$$

But the two laſt terms are equal, and therefore the two firſt, $aec = fdb$.

Cor. 1. *If two angles,* P, H, *are biſected, by the lines* PO, HO; *then the third angle* I *will alſo be biſected.* For ſince $ace = bdf$, and $a = b$, and $c = d$, or $ac = bd$, therefore $e = f$. And the contrary.

Cor. 2. *Hence alſo three perpendiculars erected on the middle of the three ſides of a triangle, interſect in one point. And the contrary.*

FIG. 11. For if two perpendiculars be erected on two ſides PH, IH, then will $b = c$, and $d = e$, whence $a = f$; therefore the perpendicular ſtands on the middle of PI.

SCHOLIUM.

If the point O be taken out of the triangle, this Prop. will hold equally true.

PROP. XVII.

12. *If four right lines be drawn from the four angles of a trapezium, to any point in it, as* O; *the products of the ſines of the alternate angles, will be equal to each other.*

In the triangle.

SPO, $a : b$ :: SO : PO
POH, $c : b$ :: PO : HO
HOI, $e : d$ :: HO : IO
IOS, $g : f$ :: IO : SO

their product

$aceg : bbdf$ :: SO.PO.HO.IO : PO.HO.IO.SO.

therefore

$aceg = bbdf$.

Cor. 1. *After the ſame manner, if lines be drawn from all the angles of any polygon whatever to a point; the products of the ſines of the alternate angles will be equal.*

13. Cor. 2. *In any trapezium, let* a, b, c, d, &c, *be the ſines of the angles, as in the figure; then*

As $bbd + ace$:
$bbd - ace$ ::

Tan. half the ſum of the angles $\frac{ISH + SIP}{2}$: FIG. 3.

Tan. half the difference $\frac{ISH - SIP}{2}$.

For ſince $aceg = bbdf$, therefore $ace : bbd :: f : g$ :: SO : IO, and by compoſition and diviſion, $bbd + ace : bbd - ace :: IO + SO : IO - SO ::$ (by Prop. VI) tan. $\frac{ISO + SIO}{2}$: tan. $\frac{ISO - SIO}{2}$.

SECT. II.

The ſolution of all the caſes of plain triangles.

FROM the foregoing propoſitions, all the caſes of plain triangles may be reſolved. Every triangle has 6 parts, 3 ſides and 3 angles; and any three things being given (except the 3 angles) the other 3 may be found; but if only 3 angles are given, there may be found an infinite number of triangles, that will have theſe three angles. I ſhall here give the ſolution of right-angled triangles three different ways, *viz.* arithmetically, logarithmically, and algebraically: and that of oblique triangles, logarithmically and algebraically. For it is needleſs to ſpend time in ſolving theſe arithmetically; ſince any oblique-angled triangle is divided into two right ones by a perpendicular, and then they are reſolved by the caſes of right-angled triangles; this method is true to 3 or 4 places of figures, and is ſufficient in common caſes. Alſo all the caſes in obliques (except the laſt) may be reſolved by the rules of right-angled triangles; but with a deal more labour; by letting fall a perpendicular from the end of a given ſide, or oppoſite to a given angle.

I have alſo omitted their ſolution by projection and inſtrumentally, as being only approximations to the truth; and they are beſides ſo eaſy in themſelves, to any body that can but handle a pair of compaſſes, as to need no particular explication. I ſhall only obſerve in general, with regard to Gunter's ſcale, &c, that any ſimple proportion may be wrought on it by this *general rule.*

Extend the compaſſes from the firſt term to one of the means, on its proper line; that extent, ſet the ſame way, upon its proper line, will reach from the other mean to the fourth term required; where radius is the ſine of 90°, or tan. 45°. If, in extending upon the tangents, the compaſſes reach beyond the end, ſet it ſo far back as it reaches over.

Now according to the different variety of things given and ſought, the ſolution of triangles is divided into ſeveral caſes as follows; in which I have only put the eaſieſt ſolutions. They that require more may conſult the propoſitions, from whence they are derived; the numerical operations are performed by *this rule.*

Add the logarithms of the ſecond and the third terms together, from which ſubtract the logarithm of the firſt term, and the remainder is the logarithm of the fourth term ſought, where radius is 10.

But in algebraic ſolutions you muſt uſe natural ſines, tangents, &c; and they muſt be actually multiplied and divided; and here radius is 1.

Note, In right-angled-triangles, if one acute angle is given, the other is found by ſubtracting this from 90.

And in oblique triangles, if 2 angles be given, the third is found by ſubtracting their ſum from 180. Or ſubtracting one from 180 gives the ſum of the other two.

In ſolving any of the caſes, it will be proper for diſtinction's ſake to mark what is given with a daſh (/), and what is ſought with a cypher (o).

Right-angled plain triangles.

CASE I.

FIG. 14. An angle C and the hypothenuſe BC, being given; to find a ſide, BA.

I. *Arithmetically without tables.*

Let D = degrees in the leſſer angle; then

$\frac{57\cdot3}{D} + \frac{3D}{1000} : 1 ::$ hyp. BC : leſſer ſide CA.

or $1 : 1 - \frac{1\frac{1}{2}DD}{10000} ::$ hyp. BC : greater ſide BA.

II. *Logarithmically, by the table of artificial ſines*, &c.

Rad : hyp. BC :: s. an angle C : op. ſide BA.

III. *Algebraically, by the table of nat. ſines*, &c,

Let hyp. BC $= h$, nat. ſine of C $= s$, rad $= 1$. then

$1 : h :: s : sh =$ BA.

CASE II.

14. Given an angle B or C, and a leg; to find the hypothenuſe, BC.

I. *Arithmetically.*

Let D = degrees in the leſſer angle, then

$1 : \frac{57\cdot3}{D} + \frac{3D}{1000} ::$ leſſer ſide AC : hyp. BC.

or $1 - \frac{1\frac{1}{2}DD}{10000} : 1 ::$ greater leg BA : hyp. BC.

II. *Lo-*

II. *Logarithmically.*

FIG.

S. angle B : oppoſite ſide CA : : rad : hyp. BC. 14.

III. *Algebraically.*

Let $s = \text{S.B}$, $p = \text{AC}$, then $\frac{p}{s} =$ hyp. BC.

CASE III.

Given an angle, and a leg CA; to find the other leg, BA. 14.

I. *Arithmetically.*

Find the hyp. BC, by Caſe II, then $\sqrt{\text{BC}^2 - \text{CA}^2} = \text{BA}$.

II. *Logarithmically.*

Rad : one leg CA : : tan. angle C : op. ſide BA.

III. *Algebraically.*

Let $\text{S.C} = s$, $\text{S.B} = c$, tan. $\text{C} = t$, $\text{CA} = p$, then

$pt = \text{BA}$, or $\frac{ps}{c} = \text{BA}$.

CASE IV.

Given the hyp. BC, and a leg CA; to find an angle B, or C. 14.

I. *Arithmetically.*

Find the other leg, BA, by Caſe V, then

As hyp. BC + ½ the longer leg BA : ſhorter leg CA : : 86 : op. angle B; and B taken from 90 gives C.

II. *Logarithmically.*

FIG. 14. As hyp. BC : rad :: given leg CA :: s. op. angle B, and B taken from 90 gives C.

III. *Algebraically.*

Let hyp. BC $= b$, leg CA $= p$, then $\frac{p}{b} =$ s.B, and $90 - \text{B} = \text{C}$.

CASE V.

14. Given the hypothenuſe BC, and a leg CA; to find the other leg BA.

I. *Arithmetically.*

$$\sqrt{\text{BC}^2 - \text{CA}^2} = \text{BA}.$$

II. *Logarithmically.*

Take the *ſum* and *difference* of the hyp. BC, and given leg AC, add their logarithms together; *half the ſum* is the logarithm of the ſide required, AB.

III. *Algebraically.*

Let BC $= b$, CA $= p$, then $\sqrt{bb - pp} =$ BA.

Or let BC + AC $= z$, BC — AC $= d$, then $\sqrt{dz} =$ BA.

CASE VI.

14. The legs AB, AC, being given; to find an angle, B.

I. *Arithmetically.*

Find the hyp. BC by Caſe VII; then, as hyp. BC + half the longer leg, BA : leſſer leg CB :: 86 : angle op. B.

II. *Lo-*

II. *Logarithmically.*

As one leg AB : rad :: other leg AC : tan. B, the op. angle. FIG. 14.

III. *Algebraically.*

Let AB $= b$, AC $= p$, then $\frac{p}{b} =$ tan. B, or $\frac{p}{\sqrt{bb+pp}}$ $=$ S.B.

CASE. VII.

The legs AB, AC being given; to find the hypothenuſe BC. 14.

I. *Arithmetically.*

$\sqrt{AB^2 + AC^2} = BC$.

II. *Logarithmically.*

Find the angle B by Caſe VI, and then the hyp. BC by Caſe II.

III. *Algebraically.*

Let AB $= b$, AC $= p$, then $\sqrt{bb+pp} =$ BC.

Oblique

Oblique plain triangles.

CASE I.

FIG. 15. 16. 17. Given two angles A, B, and an oppoſite ſide, FB; to find the other oppoſite ſide AF.

I. *Logarithmically.*

S. one angle A : *op. ſide* FB : : *S. the other angle* B : *to its oppoſite ſide* AF.

II. *Algebraically.*

Let $a =$ S. A, $s =$ S. B, FB $= d$, then $\frac{sd}{a} =$ AF.

Alſo put $n =$ *coſ.* A, $m =$ *coſ.* B. *Then* $am + ns = s$. F, *and* $\frac{am + ns}{a} d =$ AB. *Alſo* $\frac{ab}{am + ns} =$ BF.

CASE II.

15. 16. 17. Two ſides AF, FB, and an oppoſite angle B being given; to find the other oppoſite angle A.

I. *Logarithmically.*

One ſide AF : *S. its op. angle* B : : *other ſide* FB : *S. of its op. angle* A.

If the given angle B is obtuſe, or if its oppoſite ſide AF be greater than the other ſide FB, then angle A ſought is acute; otherwiſe it is doubtful: and the ſum of A and B taken from 180, gives F.

II. *Algebraically.*

Let $s =$ S. B, AF $= c$, FB $= d$, then $\frac{sd}{c} =$ S. A.

Alſo put $m = cof.\ B$, *and then* $\frac{msd + s\sqrt{cc - ssdd}}{c}$ FIG.
$= s.\ F.$

CASE III.

Two ſides AF, FB, and an oppoſite angle B, given; to find the third ſide AB. 15. 16. 17.

I. *Logarithmically.*

Find the angle A by caſe the ſecond, and then the angle F will be had. Then find AB by Caſe I.

II. *Algebraically.*

Let $AF = c$, $BF = d$, $s.\ B = s$, $cof.\ B = m$. *Then* $dm \pm \sqrt{cc - ssdd} = AB$; + if ∠ A is acute, and — if obtuſe. Where note, if B be obtuſe, *m* will be negative.

CASE IV.

Two ſides AF, FB, and the included angle F, being given; to find the other angles, A, B. 15. 16. 17.

I. *Logarithmically.*

As ſum of the ſides $AF + FB$: *their diff.* $AF - FB$: : *tan. half the ſum of the op. angles* $\frac{B + A}{2}$: *tan. of half their diff.* $\frac{B - A}{2}$; *then* $\frac{B + A}{2} + \frac{B - A}{2} = B$ *the greater angle, and* $\frac{B + A}{2} - \frac{B - A}{2} = A$ *the leſſer angle.*

II. *Al-*

II. *Algebraically.*

FIG. *Let* AF $= c$, BF $= d$, $x =$ s. F, $y =$ *coſ.* F. *Then* $\frac{dx}{c - dy} =$ *tan.* A, *and* $\frac{cx}{d - cy} =$ *tan.* B.

Or $\frac{c - dy}{\sqrt{dd + cc - 2dcy}} =$ *coſ.* A, *and* $\frac{d - cy}{\sqrt{dd + cc - 2dcy}}$ $=$ *coſ.* B.

CASE V.

15. Given two ſides AF, FB, and the contained angle
16. F; to find the third ſide AB.
17.

I. *Logarithmically.*

Find A *or* B *by Caſe* IV, *then* AB *by Caſe* I.

II. *Algebraically.*

Let AF $= c$, BF $= d$, $y =$ coſ. F, $v =$ verſ. F. Then $\sqrt{cc + dd - 2cdy} =$ AB. Or $\sqrt{\overline{c - d}^2 + 2cdv}$ $=$ AB.

See alſo the corollaries to Prop. VII.

CASE VI.

18. Three ſides given, to find an angle F.

I. *Logarithmically.*

Let fall a perpendicular BD, on a ſide adjoining to the angle ſought, then as baſe AF : ſum ſides AB + BF : : their diff. AB — BF : AD — DF the diff. ſegments, or the alternate baſe; then $\frac{1}{2}$ baſe AF + $\frac{1}{2}$ diff. = greater ſegment, and $\frac{1}{2}$ baſe AF — $\frac{1}{2}$ diff. = leſſer ſegment. Then there is given BF, DF to find F, by Caſe IV of right-angled triangles. Alſo ſee Prop. VII and cor.

II. *Al-*

II. *Algebraically.*

FIG. 18.

$$\text{Let } AF = c,\ BF = d,\ AB = b,\ AF \backsim BF = n, \left\{ \begin{array}{l} \text{then } \dfrac{cc+dd-bb}{2cd} = \text{coſ. } F. \\ \text{or } \dfrac{b^2-n^2}{2dc} = \text{verſ. } F. \end{array} \right.$$

See alſo the corollaries to Prop. VII, *particularly Cor.* 3.

SCHOLIUM.

In the ſolutions of plane triangles there are three caſes, where the thing ſought requires two operations; 1. when two legs of a right-angled triangle are given to find the hypothenuſe; 2. when two ſides and the included angle, in an oblique triangle, are given, to find the third ſide; 3. when two ſides and an angle oppoſite are given, in an oblique triangle, to find the third ſide.

The actual ſolution of all the caſes of plain triangles in numbers.

Right-angled triangles.

CASE I.

FIG. 57. Given the ∠ C = 52 13, hyp. BC = 316, to find BA.

I. *Arithmetically.*

Let D = ∠ B = $37\frac{47}{60}$ = 37.78 in decimals, and $\frac{57.3}{D} = 1.516$, and $\frac{3D}{1000} = .113$, and $\frac{1\frac{1}{2}DD}{1000} =$ .214, and 1.516 + .113 = 1.629, and 1 — .214 = .786; therefore 1.629 : 1 : 316 : 194 = CA,
and 1 : .786 : : 316 : 248 = BA.

II. *Logarithmically.*

Rad. . . .	10.0000000
BC (316) ——	2.4996871
S.C (52 13) —	9.8978103
AB = 249.7 . . .	2.3974974

CASE II.

58. Given the ∠ B = 37 47, CA = 194. To find BC.

I. *Arithmetically.*

$\frac{57.3}{D} + \frac{3D}{1000} = 1.629$ as in the laſt, then
1 : 1.629 : : 194 : 316 = BC.

II. *Lo-*

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II. *Logarithmically.* FIG. 58.

s.B (37 47) —— 9.7872317
CA (194) —— 2.2878017
Rad. — —— 10.
Hyp. BC, 316.6 . . . 2.5005700

CASE III.

Given the ∠ C = 52 13, CA = 194; to find BA. 59.

I. *Arithmetically.*

Find BC = 316, by Caſe II. Then
from the ſquare of BC = 99856
ſubtract the ſquare of CA = 37636
extract the ſquare root of 62220
which will be —— 249 = BA.

II. *Logarithmically.*

Rad. —— —— 10.
CA (194) —— 2.2878017
Tan. C (52 17) — 10.1116225
BA, 250.8 . . . 2.3994242

CASE IV.

There is given the hyp. BC = 316, and CA = 194; to find the angle B. 60.

I. *Arithmetically.*

From the ſquare of BC 99856, take the ſquare of CA 37636, the ſquare root of the remainder (62220) is 249 = BA; and 316 + (½ BA) 124½ = 440½, then

As 440½ : 194 :: 86 : 37.87 in decimals = 37 52.

II. *Lo-*

FIG.
60.

II. *Logarithmically.*

BC (316) —— 2.4996871
Rad, — — 10.
CA (194) — 2.2878017
S.B, 37 52 . . . 9.7881146

CASE V.

61. Given the hypothenuſe BC = 316, and CA = 194; to find BA.

I. *Arithmetically.*

From the ſquare of BC = 99856
take the ſquare of CA = 37636
the ſq. root of the remaind. 62220
extracted will be . . . . 249 = BA

II. *Logarithmically.*

BC 316
CA 194
Sum 510 log. 2.7075702
Diff. 122 log. 2.0863598
4.7939300
half 2.3969650 = 249.4 = BA.

CASE VI.

The legs AB = 249, and AC = 194 being given; to find the angle B.

I. *Arith-*

I. *Arithmetically.*

FIG. 62.

To the ſquare of AB = 62001
add the ſquare of AC = 37636
The ſum is 99637
Its ſquare root 315.6 = BC,
add ½ AB 124.5
440.1
then 440.1 : 194 :: 86 : 37.9 in decimals, or
= 37 54 = ∠ B.

II. *Logarithmically.*

AB (249) — — 2.3961993
Rad. — — — 10.
AC (194) — — 2.2878017
Tan. B, 37 55 . . . 9.8916024

CASE VII.

The legs AB = 249, AC = 194, being given; to find the hyp. BC. 63.

I. *Arithmetically.*

To the ſquare of AB = 62001
add the ſquare of AC = 37636
Sum 99637
Its ſquare root 124.5 = BC.

II. *Logarithmically.*

Find the angle B = 37 55, by Caſe VI; then find the hyp. BC = 315.7, by Caſe II.

Oblique triangles.

CASE I.

FIG. 64. Given two angles A = 31 07, and B = 115 24; BF = 97; to find AF.

S.A (31 07)	—	9.7133077
FB (97)	— —	1.9867717
S.B (115 24)	—	9.9558490
		11.9426207
AF = 169.6	. . .	2.2293130

CASE II.

65. Given AF = 320, FB = 468, and B = 32 15, to find the ∠ A.

AF (320)	—	2.5051500
S.B (32 15)	—	9.7272276
FB (468)	—	2.6702458
		12.3974734
S.A { 51 18 or 128 42 }		9.8923234

Since AF is leſs than B, the angle A may be either acute or obtuſe.

CASE III.

66. Given { AF = 320, B = 32 15; to find AB.
FB = 468.

Find the angle A = 51 18 or 128 42, by Caſe II; then we have F = 96 27 or 19 03.

Then by Caſe I find AB = 595.9 or 195.7; the ∠ A and ſide AB being ambiguous, becauſe AF (op. to the given angle) is leſs than FB. FIG. 66.

CASE IV.

Given $\left\{\begin{array}{l}AF = 376\\ FB = 311\\ F = 123\ 10\end{array}\right\}$ to find the angles A, B. 67.

AF =	376		180
FB =	311		123 10
Sum	687	Sum op. ∠s	56 50
Diff.	65	½ ſum	28 25

687 ——	2.8369567		then
65 ——	1.8129133	½ ſum	28 25
Tan. 28 25 ——	9.7332566	½ diff.	2 56
	11.5461699		31 21 = B
Tan. ½ diff. (2 56)	8.7092132		25 29 = A.

CASE V.

Given $\left\{\begin{array}{l}AF = 376\\ FB = 311\\ F = 123\ 10\end{array}\right\}$ required AB. 68.

By Caſe IV find A = 25 29.
Then AB will be found by Caſe I = 605.1.

CASE VI.

Given $\left\{\begin{array}{l}AF = 376\\ FB = 311\\ AB = 605\end{array}\right\}$, required ∠ F. 69.

FIG. 69. On AF let fall the perpen. BD, then

AB = 605
FB = 311
Sum 916
Diff. 294

AF (376)	——	2.5751878	then
Sum (916)	——	2.9618955	½AF = 188
Diff. (294)	——	2.4683473	½alt. = 358.1
		5.4302428	Diff. = 170.1
Alternate baſe (716.2)		2.8550550	= FD.

Then in the right ∠ triangle FDB, BF, FD are given, to find ∠ F, by Caſe IV of right ∠ triangles; then F = 56 50, and ∠ AFB = 123 10, becauſe, the alternate baſe being greater than the true baſe, the perp. BD falls without the triangle.

THE
ELEMENTS
OF
TRIGONOMETRY.

BOOK III.

The doctrine of the ſphere and ſpherical trigonometry.

DEFINITIONS.

1. *Spherical trigonometry*, teaches the properties of ſpherical triangles, and how to calculate their ſides and angles.

2. *A circle of the ſphere* is that which is made by a plane cutting the ſurface of the ſphere. If the plane paſs thro' the center, it is a *great* circle: if not, it is a *leſſer* circle.

3. *The pole of a circle* is a point on the ſurface of the ſphere equidiſtant from every point of the circle. Every circle has two poles, diametrically oppoſite to each other.

4. *A ſpherical angle* is the inclination of two circles of the ſphere to one another, interſecting in a point called the angular point.

5. *A right angle* is 90 degrees; an *acute* angle is leſs, and an *obtuſe* angle greater than a right one.

6. *A ſpherical triangle* is made upon the ſurface of the ſphere, by the interſection of three great cir-

cles. If one angle be right, it is called a *right-angled* ſpherical triangle. If one ſide be a quadrant, it is called a *quadrantal* triangle. If no angle or ſide be 90°, it is an oblique triangle.

7. In a right-angled triangle, the ſide oppoſite to the right angle is called the *hypothenuſe*; the other two are called *legs* or ſides.

8. In a right-angled or quadrantal triangle, the five circular parts (ſetting aſide the right angle or quadrantal ſide) are, the two parts adjoining on each ſide the right angle or quadrantal ſide, and the complements of the three parts which are furtheſt diſtant from it. The *middle part* is that which is equidiſtant from other two; which two if they adjoin to the middle part, are called *extremes conjunct*; and if they be remote from it, are called *extremes disjunct*.

9. Sides and angles are ſaid to be of the *ſame affection*, when both are greater, or both leſſer, or both equal to 90°. And they are of *different affection*, if one is greater and another leſs than 90°.

10. *Like or ſimilar arches* of different circles are thoſe that contain the ſame number of degrees.

SECT. I.

The properties of ſpherical angles and arches.

PROP. I.

The ſection of a ſphere and a plane is a circle.

LET BIC be the ſection, and let the plane of the great circle ABCD be drawn through the center E of the ſphere, perpendicular to the plane BIC, and let BHC be their common interſection. Draw EH perpendicular to BC, then BH = HC. Take any point I in it, and draw HI, IC; and alſo BE, CE. Since EH is perpendicular to BC, and in the perpendicular plane ABCD, therefore it is perpendicular to the plane BIC, and therefore to HI. But BE, CE, IE are equal radii of the ſphere; therefore in the right-angled triangles BEH, IEH, $BH^2 + HE^2 = BE^2 = IE^2 = HI^2 + HE^2$, therefore $BH^2 = IH^2$, and BH = HI, therefore BIC is a circle whoſe center is H. FIG. 19.

Cor. 1. *If a great circle* ABCD *be perpendicular to any other circle* BIC, *it paſſes thro' its poles.*

For ſince it is proved that H is the center of the circle BIC, and EH perpendicular to it, produce EH to P, which will be in the plane of the circle ABD, becauſe EH is in it. Draw the great circle PI; then ſince the ſines BH, HI, HC are equal, therefore the arches belonging to them BP, IP, CP are equal, therefore P is the pole. And if PE was produced to the other pole, that pole will alſo be in the plane of the circle ABC.

FIG. 19. Cor. 2. *If a great circle* ABC *paſſes thro' the pole* P *of another circle, it cuts it at right angles and into two equal parts.*

For draw PE to cut the circle in H, then is H the center, and PH perpendicular to its plane, and conſequently EH and the plane ABCD are perpendicular to it. And ſince the center H is in BC, the common interſection of the planes; therefore BIC is a ſemicircle.

Cor. 3. *The angle made by two leſſer circles (or any circles), is equal to the angle made by two great circles, drawn from their poles to that angle.*

For both circles make right angles with their reſpective ſmall circles (by Cor. 2); and taking away what is common, there remains the angle of the great circles = angle of the ſmall circles.

PROP. II.

20. *All great circles of the ſphere cut one another into two equal parts.*

For the common ſection of their planes is the diameter of the ſphere, and conſequently their ſegments are ſemicircles.

Cor. 1. *Any ſide of a ſpherical triangle is leſs than a ſemicircle.*

For ſince ADC or ABC is a ſemicircle, therefore in the triangle ADB, AD or AB is leſs than a ſemicircle.

Cor. 2. *If two ſides of a ſpherical triangle* ABD *be produced, till they interſect in* C, *each will become a ſemicircle.*

PROP. III.

21. *If one or more circles interſect another circle in one and the ſame point* C; *the ſum of the angles* ACE, ECD *is equal to two right angles.*

Let the arch EC interſect AD in C, and draw CB perpendicular to AD; then ACB + BCD = two right angles;

gles; but ACE + ECB = ACB, and ECD — ECB = CD, therefore ACE + ECD = ACB + BCD = two right ngles. FIG. 21.

Cor. 1. *All the angles made about one point are = four right angles.*

Cor. 2. *When one circle croſſes another, the vertical angles a, c are equal.*

For $a+b=$ two right angles, and $b+c=$ two right angles, therefore $a+b=b+c$ and $a=c$.

PROP. IV.

A ſpherical angle made by two great circles is meaſured by the arch of a great circle intercepted between the ſides, and deſcribed at a quadrant's diſtance from the angular point. 22.

Let AB = AD = 90°, and draw the arch BD, and from the center E draw BE, DE; alſo draw the tangents AT, AS, to the arches AB, AD; then the ∠ TAS = ſpherical angle BAD, and AEB, AED will be two right angles; but TAE, SAE are right angles, therefore TA, SA are ∥ to BE, DE, and ∠ TAS = BED. But the arch BD is the meaſure of the angle BED or TAS, and therefore of the ſpherical angle BAD.

Cor. 1. *The angles* BAD, BCD, *made at oppoſite points of the ſemicircle, are equal. For the* ∠ BED *or arch* BD *is the meaſure of both.*

Cor. 2. *The diſtance of the poles of two circles is equal to the angle of their inclination.*

For ſince AE is ⊥ to the plane BDE, this plane is ⊥ to both the planes ABC, ADC; therefore the poles are in the circle BD, as ſuppoſe at P, Q. Then BP = 90° = DQ, and ſubtracting DP from both, then BD = PQ.

Cor.

FIG. 22. Cor. 3. *Two great circles* ABC, ADC *paſſing thro' the poles of another great circle* BD, *will cut all the parallels to* BD, *as bd, into ſimilar arches.*

For AC is ⊥ to *bed*, and *e* the center of *bd*, therefore ∠ BED = *bed*, or the arch *bd* ſimilar to BD.

Cor. 4. *Hence an angle made by two great circles of the ſphere, is equal to the angle of inclination of the planes of theſe great circles.* For ∠ BED or *bed* = ∠ TAS or BAD.

Cor. 5. *Hence alſo if bd be a parallel to the great circle* BDP; *it is as radius : coſine of the parallel's diſtance from its great circle : : ſo any arch of the great circle : to a ſimilar arch of the parallel.*

For BE : BD : : *be* : *bd*.

PROP. V.

23. *If any two circles* AEB, AFB *interſect one another on the ſphere, they make the oppoſite angles* A *and* B *equal, and the oppoſite parts* AFE, BFE *ſimilar.*

Let AB be the common ſection of their planes; and in the planes of their reſpective circles, draw the tangents AC, BC to the circle AFB; and AD, BD to the circle AEB, and join CD. Then in the triangles ACD, BCD, AC = BC, being tangents to the ſame circle, and for the ſame reaſon AD = BD, and CD is common; therefore the triangles are equal and ſimilar, and the ∠ CAD = CBD, and therefore the ſpherical angles A and B equal to them, are alſo equal to one another.

Likewiſe the oppoſite parts of the figure AFBEA are ſimilar, being bounded by parts of the ſame circles, having the ſame poſition.

Cor. *Hence any two like arches* OI, SV, *drawn after a like manner, will be equal. That is, if* AO = BS,

and AI = BV, *or* ∠ AOI = ∠ BSV, *then* OI = SV: FIG.
For then the triangles AOI and BSV are ſimilar and 23.
equal.

SCHOLIUM.

Though the oppoſite parts of the figure AFE and BFE are ſaid to be ſimilar; yet if the angle A be applied to B, the triangles AFE, BFE will not coincide. Or if the baſe FE of one be laid on the baſe EF of the other, the triangular figures AFE, BFE will lie contrary ways, one to the right, the other to the left hand.

PROP. VI.

If two great circles HO, HR *have the ſame inclination to a third* EQ; *the arches of all the parallels* (AP) *to the third, intercepted by the other two, will be ſimilar arches.* 24.

Let the great circle EC cut the circles HO, HR at equal angles in E and Q. Thro' the poles of EC draw the great circles PB, AC. Then in the triangles PEB, AQC, ∠ E = ∠ Q, and the angles at B and C are right, and PB = AC; therefore if the angle E be laid upon the angle Q, then will P fall ſomewhere in the arch PA (becauſe PB = AC); and ſomewhere in the arch QH (becauſe ∠ E = ∠ Q), therefore P falls upon A, their point of interſection; and conſequently B upon C (otherwiſe A would be the pole of CQ); therefore BE = CQ, and PE = AQ, therefore EQ = BC; but BC is ſimilar to AP (by Cor. 3 Prop. IV), therefore EQ and AP are ſimilar.

Cor. *The ſame things ſuppoſed, the arches of the two inclined great circles intercepted between any two parallels to the third, are equal,* PE = AQ.

SCHOL. See Prop. XVI.

PROP.

PROP. VII.

FIG. 25. *If a plane* EPQ *be ſuppoſed to be drawn perpendicular to the planes of two equal circles* HO, EQ; *and any other plane* PDKN *be drawn thro' the two poles* P, N, *which are equidiſtant from their reſpective circles, and to revolve about* PN: *then theſe planes will intercept equal arches of theſe circles,* DO, KQ.

For let N be the pole of HO, and P of EQ, and NO = PQ, then by cor. Prop. v, ∠ NOH or NOD = PQE or PQK; and ∠ POD = NQK, and NQ = PO, and by Prop. v, ∠ N = P. Therefore the triangles POD and NQK are ſimilar and equal, and therefore arch DO = KQ. And likewiſe arch HD = EK.

PROP. VIII.

26. *Of ſeveral arches of great circles drawn from the ſame point* A *to any other circle* CED; *the greateſt* AD *is that which paſſes thro' the pole* P, *and the nearer to this as* AB *is greater than that which is further off as* AE. *And the leaſt is* AC, *the remainder of that which paſſes thro' the pole.*

Let AS be perpendicular to the plane of the circle CED, O its center, CSOD its diameter. Draw the right lines SB, SE, AD, AB, AE, AC. Then you have the right-angled plane triangles ASD, ASB, ASE, ASC, wherein the perpendicular AS is common. But (Geom. IV 4) the baſe SD paſſing thro' the center is the greateſt, and SB greater than SE, and SE greater than SC, which is the leaſt.

Therefore the hypothenuſe AD is the greateſt, AC the leaſt, and AB greater than AE. But theſe are the cords of arches, and to greater cords belong greater arches, conſequently the arch AD is the greateſt, arch AB (nearer to AD) is greater than arch AE (further from it); and arch AC the leaſt of all. FIG. 26.

Cor. 1. *If two ſides in two ſpherical triangles* APE, APB *be reſpectively equal; that which hath the greater included angle, hath the greater baſe: and the contrary.* 27.

For, about the pole P, at the diſtance PE or PB (the greater *ſide* in the two triangles), deſcribe the circle EBD, and from A thro' P draw the great circle APD. Then by this Prop. if BD be leſs than ED, or $\angle$ APB greater than APE; then is AB greater than AE.

And if AB be greater than AE, then AB is nearer to D than AE, $\angle$ APB greater than APE.

Cor. 2. *The perpendicular let fall on the baſe of a ſpherical triangle, is either greater than either ſide, or leſs than either ſide.*

For it paſſes through the pole of the baſe, and is therefore either the longeſt or ſhorteſt line drawn from the vertex of the triangle.

Cor. 3. *And therefore when the perpendicular falls without the triangle, either the greater or leſſer perpendicular may be eſteemed the perpendicular upon the baſe.*

Cor. 4. *Hence if the perpendicular fall without, as in the triangle* BAE, *the greater perpendicular* AD *lies next the greateſt ſide* AB; *and the leaſt perpendicular* AC *next the leaſt ſide* AE. 28.

But when the perpendicular falls within; if it is leſs than a quadrant, as AC (in the triangle AEF), then it lies nearer the leſſer ſide AE, and the ſegment CE is leſs than CF, and $\angle$ CAE leſs than CAF. But if it is greater than a quadrant, as AD (in the triangle ABG), then it lies neareſt the greater ſide AG, and then it makes the leſſer ſegment of the baſe

FIG. 28. baſe (DG), and leſſer vertical angle (DAG), next the greater ſide AG. And the greater ones next the leſſer ſide AB.

Cor. 5. *The perpendicular falling within, or the neareſt perpendicular falling without, is of the ſame affection, as half the ſum of the ſides.*

For in the triangle EAF, AC is leſs than a quadrant, and ſince $AF < AG$, therefore $AE + AF < EG$ or 180, and $\frac{AE + AF}{2} < 90$. But in the triangle GAB, AD is > a quadrant; and ſince $AG > AF$, therefore $BA + AG > BF$ or 180, and $\frac{BA + AG}{2} > 90$.

Again, in the triangle BAE, where the perpendicular AC is leſs than a quadrant, and $CE < CF$; then by this Prop. $AE < AF$, and $AE + AB < BF$ or 180°, and $\frac{AE + AB}{2} < 90$. But in the triangle GAF, where the perpendicular AD (greater than a quadrant) is neareſt, or $DB > DG$, or $CF > CE$, then $AF > AE$, and $AF + AG > GE$ or 180, and $\frac{AF + AG}{2} > 90$.

Cor. 6. *Hence, when the perpendicular falls within, or you take the neareſt perpendicular falling without; then if half the ſum of the ſides be leſs than a quadrant, the leſſer ſegment, and leſſer vertical angle, adjoins to the leſſer ſide; but if half the ſum of the ſides be greater than a quadrant, the leſſer ſegment and leſſer vertical angle adjoins to the greater ſide.*

SECT. II.

The affections of ſpherical triangles.

PROP. IX.

Every triangle ABC *hath (by producing its ſides) another triangle abc, on the oppoſite ſide of the globe, ſimilar and equal to it.* FIG. 29.

For the point a is oppoſite to A, or a ſemicircle diſtant from it; and b is oppoſite to B, and c to C. Therefore $ab = \mathrm{AB}$, $bc = \mathrm{BC}$, and $ac = \mathrm{AC}$. Likewiſe, by Cor. 1 Prop. IV, $\angle a = \angle \mathrm{A}$, $b = \mathrm{B}$, and $c = \mathrm{C}$; therefore the triangle abc is equal and ſimilar to ABC.

Cor. 1. *Hence any triangle* ABC, *by producing its ſides, hath oppoſite to every angle thereof, another triangle of the ſame baſe and oppoſite angle with the former; and the other parts the ſupplements thereof.*

Thus in the triangle B*ca*, $\angle \mathrm{B} = \angle \mathrm{ABC}$, $ac = \mathrm{AC}$, $a\mathrm{B} =$ ſup. AB, $c\mathrm{B} =$ ſup. CB, $\angle ca\mathrm{B} =$ ſup. C*a*B or A, and $\angle ac\mathrm{B} =$ ſup. C.

Cor. 2. *Any triangle* ABC, *has adjoining to every ſide thereof another triangle, having the ſame ſide and oppoſite angle, and the other parts the ſupplements thereof.*

Thus in the triangle AB*c*, AB and $\angle c$ is the ſame with AB and $\angle \mathrm{C}$; B*c*, A*c* are the ſupplements of BC, AC; $\angle c\mathrm{BA} =$ ſup. CBA, and $\angle c\mathrm{AB} =$ ſup. CAB. Likewiſe there is the triangle *a*BC adjoining to CB; and the triangle A*b*C adjoining to AC.

Cor. 3. *If all the ſides of a right-angled triangle* ABF, *be produced to quadrants,* ABC, AFD, BFE, *and* 41.

DE,

FIG. 41. DE, AE, *be drawn. Then the right-angled triangle* FDE *will be complemental to the triangle* ABF.

In the complemental triangles ABF, EDF, the parts in one are the complements of the parts in the other; only the oppoſite angles at F are equal.

For A, E are the poles of CDE, ABC; therefore CDE, ABC, are quadrants. Whence FD = complement of AF, EF = complement of FB, ED = comp. of DC or ∠ FAB. And ∠ FED = BC = comp. of AB.

SCHOLIUM.

29. Tho' the triangles ABC, *abc* are ſaid in this Prop. to be ſimilar; yet they will not coincide when applied to one another. For laying any angle *a* upon its equal A, ſo that the concavities of the triangles lie the ſame way; the ſide *ac* will fall on AB, and *ab* on AC.

PROP. X.

30. *If from the three angles of any ſpherical triangle* ABC, *as poles, you deſcribe three great circles, they will form another ſpherical triangle* DEF *by their interſections; each ſide and angle whereof will be the ſupplements of the angle and ſide oppoſite, in the given triangle.*

Produce the ſides as in the fig. then ſince B*k* or BF is a quadrant, and A*g* or AF is a quadrant, therefore F is the pole of AB; and likewiſe E the pole of AC, and D the pole of CB. Then DQ = a quadrant = ER, and DE = QR = ſup. ∠ C.

Alſo E*g* = 90° = F*p* = T*p*, and TE = *pg*, therefore FE = ſup. ET or *pg*, that is, of the ∠ CAB. Likewiſe DT = *kr* or ∠ B, and DF = ſup. ∠ B.

Alſo *r*B = 90° = A*p*, and *r*B + A*p*, or *rp* + AB = 180°, and *rp* or the ∠ F = 180° — AB = ſup. AB.

And thus A*g* = CR, and AC = *g*R, and ſup. *g*R = ſup. AC, that is, ∠ E = ſup. AC.

Laſtly, CQ = 90° = B*k*, and CQ + B*k* or *k*Q + CB = 180, and *k*Q = 180 — CB, that is, ∠ D = ſup. CB. FIG. 30.

Cor. 1. *Some three poles of the ſides of any triangle, form another triangle, wherein the angles and ſides are reſpectively equal to the ſupplements of the ſides and angles of the other; and theſe two triangles are mutually ſupplemental to each other.*

For D, E, F are the poles of the ſides of the triangle ABC, as well as A, B, C are the poles of the ſides of DEF.

Cor. 2. *The neareſt poles* D, E, T *of the ſides of any ſpherical triangle* ABC, *form another triangle, wherein the angles and ſides are reſpectively equal to the ſides and angles of the given one: excepting only the ſupplements of one ſide and its oppoſite angle in the former, will be the correſpondent angle and ſide in the latter.*

For in the triangles DFE, DTE, DE is common, and ∠ F = ∠ T; and all the reſt are the ſupplements. There are other five, or in all, ſix ſuch triangles, and all adjoining to the angles and ſides of the triangle DEF, by Cor. 1, 2 Prop. IX; but there is only one ſupplemental triangle DEF to the triangle ABC.

SCHOLIUM.

By this prop. quadrantal triangles may be reduced to right-angled ones.

PROP. XI.

In two triangles, if three ſides in one be reſpectively equal to three ſides in the other; or the three angles in the one to the three angles in the other; theſe triangles will be equal in all reſpects.

FIG. 30. *Caſe* 1. If three ſides be equal; and if the baſe of one triangle be laid on the baſe of the other, the two other ſides muſt coincide; becauſe there can be but one point from whence two given arches can be drawn to the two angles at the baſe.

Caſe 2. If three angles are equal, then by Cor. 1 Prop. x the ſides of two triangles formed by the poles of both, will be the ſupplements of the angles of the former, and therefore reſpectively equal. Therefore, by Caſe 1 the angles of theſe ſupplemental triangles are equal, and theſe angles are the ſupplements of the ſides of the former triangles by Cor. 1 Prop. x, therefore the ſides of the former triangles are reſpectively equal.

SCHOLIUM.

This demonſtration ſuppoſes the equal ſides to lie the ſame way; but if they lie contrary ways, they cannot coincide. Yet if you will ſuppoſe the convexity of one of them to lie the contrary way, then they will coincide when laid upon one another. The ſame may be applied to the following prop.

PROP. XII.

If in two triangles, there be two ſides and the included angle; or two angles and the included ſide, reſpectively equal; the two triangles will be equal in all reſpects. For,

1. If one ſide in one triangle be laid upon its equal in the other triangle; then, by reaſon of the included angle being equal in both, the other two ſides will coincide, and ſo the whole triangle.

2. If the equal ſide in one be laid upon that in the other, and the equal angles upon one another, the whole triangles will coincide, and will therefore be equal.

PROP. XIII.

The ſum of any two ſides of a ſpherical triangle is greater than the third ſide. FIG. 31.

Let AB be the greateſt ſide; about the pole B, thro' C deſcribe the arch CD, which will be $\perp$ to BD, alſo about the pole A thro' D deſcribe the arch DE, which will alſo be $\perp$ to AD. And ſince the two circles CD, ED, can but touch in one point D, therefore at E, C there is a ſpace EC between them. Whence, ſince $AE + CB = AD + DB$ or AB, therefore $AE + EC + CB$ or $AC + CB$ is greater than AB.

Cor. 1. *The ſum of the three ſides of any ſpherical triangle* DEF *is leſs than a circle.* 30.
For DE is leſs than $DT + TE$, and therefore $DE + EF + FD$ is leſs than $FDT + FET$ or two ſemicircles.

Cor. 2. *A great circle is the neareſt diſtance of any two points upon a ſphere.* For it is a line of the leaſt curvature that can be drawn, or deflects leaſt from a right line.

PROP. XIV.

In any triangle equal ſides are oppoſite to equal angles. 32.

1. Let $AC = BC$, let CD biſect the angle ACB, then in the triangles ACD, CDB, there are two ſides and the included angle at C reſpectively equal; therefore by Prop. XII the $\angle A = \angle B$.

2. If $\angle A = \angle B$, make $AE = BE$, then by Caſe 1 the $\angle EAB = \angle B = \angle CAB$ by ſuppoſition; therefore E falls upon C, and AC is $= BC$.

FIG. 32. Cor. 1. *An equilateral triangle is alſo equi-angular, and the contrary.*

Cor. 2. *In any iſoſceles triangle, a perpendicular let fall on the baſe, biſects it and the vertical angle: et contra.*

Suppoſe the convexity of the triangle CDB to lie the contrary way; then becauſe the angles at D are equal; if CDB be laid upon CDA, DB will fall upon DA; and becauſe $CB = CA$, and $\angle A = \angle B$ (by this prop.) therefore the point B will fall upon A; and the triangles will coincide, then will $AD = DB$, and $\angle$s at C will be equal.

Again, If $AD = DB$, or $\angle ACD = \angle BCD$; then by Prop. XII the angles at D are equal, or $CD \perp AB$.

PROP. XV.

31. *In any ſpherical triangle, the greateſt ſide is oppoſed to the greateſt angle; and the leaſt to the leaſt.*

1. Let $\angle C > B$, and make $\angle BCD = B$; then will $CD = BD$ by Prop XIV, but $CD + AD$ or $AB > AC$ by Prop. XIII. For the ſame reaſon if $B > A$, then $AC > CB$.

2. Let $AB > AC$, then angle $C > B$; for if C be equal or leſs than B, then AB is equal or leſs than CA by Caſe 1, which is contrary to the ſuppoſition.

PROP. XVI.

33. *If one ſide* DA *of a triangle* DAB *be produced, and if the ſum of the other ſides* $DB + BA$ *be greater, equal, or leſs than a ſemicircle; the internal angle at the baſe* D, *is accordingly greater, equal, or leſs than the outward oppoſite angle* BAC.

For if $DB + BA$ be $>$ $=$ or $<$ than DC, then BA is $>$ $=$ or $<$ than BC; and by Prop. XV the angle C or D is > = or < than BAC, reſpectively. FIG.

Cor. 1. *In an iſoſceles triangle, as one of the equal ſides is greater, equal, or leſs than a quadrant; the angle at the baſe is accordingly greater, equal, or leſs than a right angle.*

Cor. 2. *In an iſoſceles triangle, either ſide is of the ſame affection as its oppoſite or adjacent angle.*

PROP. XVII.

33. *In any ſpherical triangle, if the ſum of any two ſides be greater, equal, or leſs than a ſemicircle; the ſum of the oppoſite angles is accordingly greater, equal, or leſs, than two right angles.*

For by Prop. XVI, if DB + BA be greater, equal, or leſs than DC, then ∠ D is greater, equal, or leſs than BAC, and D + DAB greater, equal, or leſs than DAB + BAC, or two right angles.

Cor. 1. *In any triangle half the ſum of any two ſides is of the ſame affection, as half the ſum of their oppoſite angles.*

34. Cor. 2. *Hence it follows, that if in a triangle* ACB, *there be given two ſides* AC, CB, *and an oppoſite angle* A; *and if* CD *be made* = CB, *then you will have two triangles* ACB, ACD *containing the things given, and therefore the other oppoſite angle may be either* B *or* ADC; *and conſequently in both of them the ſum of the ſides* AC, BC, *or* AC, DC, *are of the ſame affection (with reſpect to* 180°), *as the ſum of the oppoſite angles* A *and* B, *or of* A *and* ADC. Therefore, whenever the ſum of the angles A and B are of the ſame affection as the ſum of A and the ſupplement of B, then

FIG. then it is doubtful whether the other angle B is acute or obtuſe, becauſe it may be either. But when theſe are of different affections, that value only of B (whether acute or obtuſe) muſt be taken, which, when added to A, their ſum is of the ſame affection, as the ſum of the oppoſite ſides.

35. Cor. 3. *And for the ſame reaſon, when two angles* A *and* B, *and an oppoſite ſide* CB *are given: if there be two triangles* ACB, DCB *that contain the ſame data, it will be doubtful whether the other oppoſite ſide be* AC *or its ſupplement* CD *(for* CD = CE, *the angles* A, D, E *being equal). For it may be either of them.* Therefore we muſt always take that value of AC which added to CB, their ſum may be of the ſame affection (with reſpect to 180), as the ſum of the oppoſite angles. And if both ſums (*viz.* of AC or its ſupplement, added to CB) be of one and the ſame affection; then it is doubtful. Alſo if CI be ⊥ AB, it will be ambiguous whether AI or its ſup. EI = DI be the ſegment of the baſe, and whether ACI or its ſup. ECI or DCI be the angle at the vertex.

Note, The caſes mentioned in the two laſt corol. contain the ſix ambiguous caſes of oblique triangles. For when two ſides and an angle oppoſite are given; the other three things will have different values, according as an arch or its ſupplement is taken for the other oppoſite angle. And if two angles and an oppoſite ſide be given; according as an arch or its ſupplement is taken for the other oppoſite ſide; the baſe and its oppoſite angle will have different values.

33. Cor. 4. *If in a right-angled triangle, there be given an angle and its oppoſite ſide, there will be two triangles* DBA, CBA *that have the ſame data, and therefore the*

FIG. 33. *the other three things are doubtful. Whence the hypothenuſe may be either* DB *or its ſup.* BC. If DB be the hypothenuſe, then the baſe is DA and ∠ oppoſite DBA. If BC is the hyp. then AC is the baſe and ABC the op. angle, theſe three being the ſup. of the others. And theſe are called the three ambiguous caſes in right-angled triangles; when an angle and its op. leg is given, to find the reſt.

After the ſame manner in a quadrantal triangle, if a ſide and the oppoſite angle be given; the angle oppoſite to the quadrantal ſide may be either that angle DAB, *or its ſupplement* BAC. And therefore the three things ſought are ambiguous, and each of them may be either an arch or its ſupplement.

Cor. 5. From what has been ſaid it follows, that *in two ſpherical triangles, if two ſides and an oppoſite angle, be reſpectively equal in both triangles; and the other oppoſite angles in both triangles, of the ſame kind, either both acute or both obtuſe; then theſe two triangles will be equal in all reſpects.*

And if two angles and an oppoſite ſide be reſpectively equal in both; and the other oppoſite ſides, of the ſame kind: then the triangles are equal in all reſpects.

PROP. XVIII.

In any triangle ABC, *the ſum of the three angles is greater than two right angles.* 30.

Let FDE be the ſupplemental triangle to ABC, by Prop. X; then by Cor. 1 Prop. XIII, $DF + FE + ED$ is leſs than a circle; that is, the ſupplements of A, B, and C is leſs than 2 ſemicircles, or $180 - A + 180 - B + 180 - C < 360$; ſubtract 2 ſemicircles, and then $180 - A - B - C < 0$; add

FIG. $A + B + C$, then $180 < A + B + C$, or $A + B + C > 180$.

Cor. 1. *The ſum of the three angles of any triangle is leſs than ſix right angles. For every angle muſt be leſs than 2 right angles, otherwiſe it would be one continued arch.*

33. Cor. 2. *In any triangle* DAB, *the external angle* BAC *is leſs than the ſum of the interior oppoſite angles* D *and* B.

Cor. 3. *The ſum of any two angles is greater than the ſupplement of the third angle.*

30. Cor. 4. *The ſum of any two angles — the third (or any angle and the difference of the other two) is leſs than two right angles.*

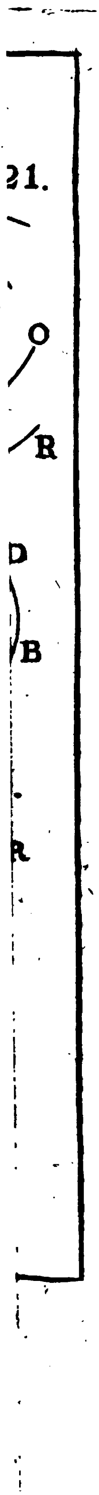
For FD is leſs than FE + ED, by Prop. XIII, that is, $180 - B$, is leſs than $180 - A + 180 - C$; that is, 180 ~~$+ A + C - B$~~ is leſs than $180 + 180$, or $A + C - B$ leſs than 180.

Cor. 5. *In a right-angled ſpherical triangle, the ſum of the oblique angles is greater than one, and leſs than three right angles. By this prop. and Cor.* 4.

PROP. XIX.

28. *If the angles at the baſe of a ſpherical triangle be of the ſame affection, the perpendicular on the baſe will fall within the triangle; if of different affection, it falls without.*

In the triangle GAB, whoſe baſe is GB and its pole P, ſince the perpendicular muſt paſs thro' the pole



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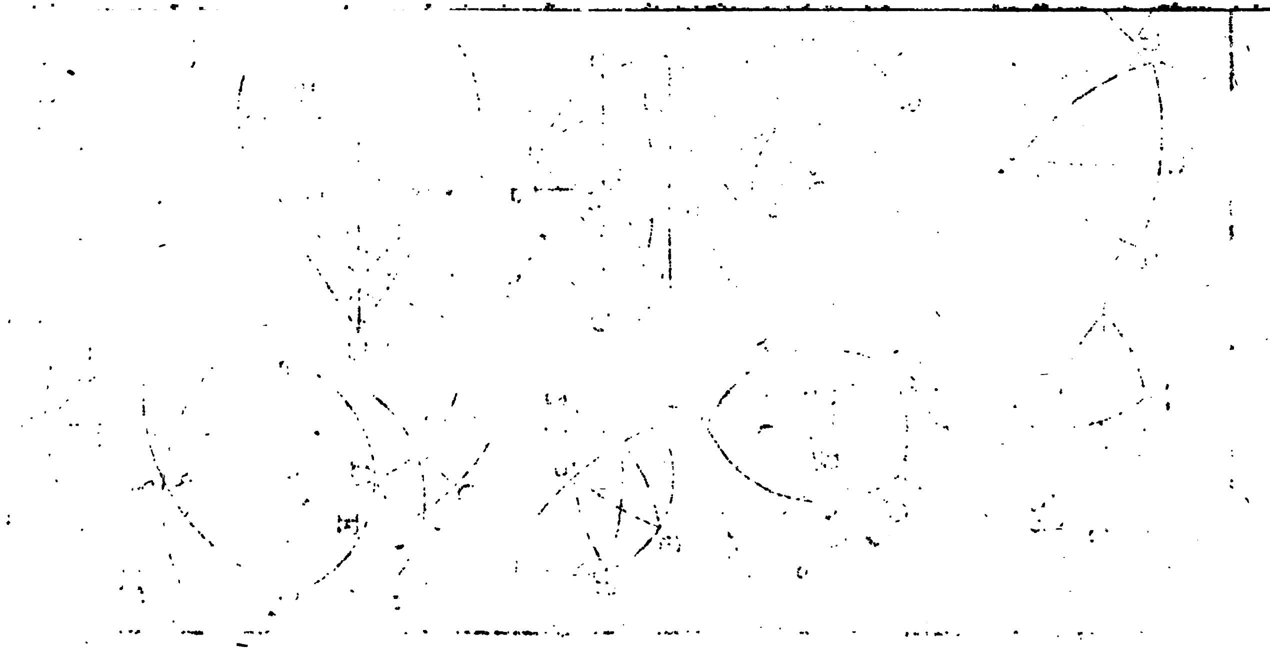

pole P, by Prop. i Cor. i, let it interſect the baſe in D, C, then the angles PGB and PBG are right, and therefore AGB and ABG are both obtuſe, when the perpendicular CD falls between them. Likewiſe the angles PFE, and PEF are both right; therefore in the triangle FAE the angles F and E are both acute, when AC falls within. But in the triangle BAE the perpendicular falls without, and the ∠ ABE is acute and AEB obtuſe. And the ſame will hold for any other triangle that can be drawn. FIG.

Cor. *If the two leaſt ſides of a ſpherical triangle,* AC, BC *be of the ſame affection; the perpendicular* CD *upon the other* AB, *will fall within the triangle.* 36.

Let AB be the greateſt ſide, make BE = BC, and AF = AC, then by Cor. 2 Prop. xvi, the ∠ BEC is of the ſame affection as CB, and AFC of the ſame affection as AC; but AC, BC are of the ſame affection, and therefore the angles EFC, FEC. Therefore by this prop. the perpendicular CD falls within the triangle ECF or ACB.

If AB be leſs than AC, CB, but greater than their ſupplements (they being of the ſame affection), the perpendicular will fall within. For this and the adjoining triangle (whoſe common baſe is AB) have the ſame perpendicular. See Cor. 2 Prop. ix.

SCHOLIUM.

Some other properties of perpendiculars may be ſeen in the corollaries to Prop. viii.

PROP. XX.

In any right-angled ſpherical triangle, each of the oblique angles is of the ſame affection as the oppoſite ſide. 37.

If

FIG. 37. If the angle at A be right, and AC leſs than a quadrant, produce AC to D, that AD may be a quadrant, and draw BD; then D is the pole of BA, and conſequently ABD a right angle, and therefore ABC is leſs than a right angle. But if AE be greater than a quadrant, then its oppoſite angle ABE is greater than the right angle ABD. And if AD be a quadrant, the angle ABD is a right angle.

On the contrary, if ABD is a right angle, then D is the pole of AB, and AD a quadrant; but if ABC be leſs or ABE greater than a right angle, then the oppoſite ſide AC will be leſs, or AE greater than the quadrant AD.

Cor. *In a right-angled ſpherical triangle, if the hypothenuſe be a quadrant, one of the ſides will be a quadrant, and its oppoſite angle a right angle.*

Let BAC be the triangle, A the right angle, the hypothenuſe BC a quadrant; AB (produced at leaſt) paſſes thro' the pole of AC; and if the quadrant CB be ſuppoſed to revolve about C, it will cut AB in the pole of AC. But B is the point where they cut, therefore B is the pole of AC; and conſequently BA, BC, are both quadrants; and A being a right angle, C will alſo be a right angle (by Prop. XIV.)

Or thus.

Suppoſe C a right angle, then will B be the pole of AC, and therefore BC, BA, are both quadrants. But if C is acute, then CB will be leſs than a quadrant, when AC is leſs; or greater when AC is greater, (ſee Prop. VIII), both contrary to the ſuppoſition. And when C is obtuſe, CB will be greater than a quadrant, when AC is leſs, and leſs when AC is greater, both alſo contrary to the ſuppoſition.

Therefore in all caſes when A is a right angle, and BC a quadrant, C will be a right angle too, and conſequently (B being the pole of AC), BA will be a quadrant. FIG. 37.

PROP. XXI.

In a right-angled ſpherical triangle BAD, *if the legs be of the ſame affection, the hypothenuſe* BA, *will be leſs than a quadrant; if of different affection, it will be greater: and the contrary.* 38.

Let D be the right angle, and produce DB, DA to C; and make DP, DE quadrants, then P is the pole of DB, and E of DA, and PB a quadrant. Then in the triangle DBA, if DB, DA be leſs than a quadrant, then BA will be leſs than the quadrant BP, by Prop. VIII. And in the right-angled triangle BAC, if BC, AC, be greater than a quadrant, then alſo BA is leſs than BP.

But in the triangle DBF, where DB is leſs and DF greater than a quadrant; as likewiſe in the triangle BCF, the hypothenuſe BF will be greater than the quadrant BP, by the ſame Prop. VIII.

On the contrary, if the hypothenuſe BA be leſs than a quadrant BP, then DB, DA are both leſs; or elſe CB, CA both greater than a quadrant. But in the triangle DBF or CBF, where the hypothenuſe BF is greater than the quadrant BP; DB, DF or elſe CB, CF are of different affections.

Cor. 1. *As the two oblique angles are of the ſame or different affection; the hypothenuſe will be accordingly leſſer or greater than a quadrant: et contra, by Prop.* XX *and* XXI.

Cor. 2. *As the hypothenuſe and one ſide (or its oppoſite angle) is of the ſame or different affection; the other ſide*

FIG. 38. *ſide (and its oppoſite angle) will be leſs or greater than a quadrant.*

Cor. 3. *As the hypothenuſe is leſſer or greater than a quadrant, each leg will be ſimilar or diſſimilar to its adjacent angle: and the contrary.*

Cor. 4. *If the hypothenuſe* AB *be leſs or greater than a quadrant, one ſide* AD *ſhall alſo be leſs or greater than a quadrant, when* DB *(the other ſide) is leſs; and the oppoſite angle* B *ſhall alſo be leſs or greater than a right angle. But the contrary will happen, when* DB *is greater than a quadrant.*

PROP. XXII.

39. *If a triangle* CIA *have one acute angle, it alſo hath one ſide leſs than a quadrant.*

Let ACI be the acute angle, and in the triangle CBA, let C be a right angle; if either A fall between P and C, or B between Q and C, the prop. is manifeſt; for then either CA or CB is leſs than a quadrant. Let therefore CB, CA be greater than quadrants, then by Prop. XXI, BA is leſs than a quadrant, therefore in the acute-angled triangle ACI, AI is ſtill leſs than a quadrant.

Cor. *If a triangle have one ſide greater than a quadrant, it alſo hath one obtuſe angle.*

For if a triangle hath one ſide greater than a quadrant, its ſupplemental triangle (by Prop. X) hath one acute angle; and therefore by this prop. it hath a ſide alſo leſs than a quadrant. And if ſo, then (by Prop. X) the former has an angle greater than a right angle.

PROP.

PROP. XXIII.

If a triangle has two ſides leſs than quadrants, it has alſo an acute angle. FIG. 39.

For in the triangle DBA, if BD, DA be leſs than quadrants, the ſum of the oppoſite angles is leſs than 2 right angles, by Prop. XVII; therefore at leaſt one of the angles is leſs than a right angle.

Cor. *If a triangle* ACB *has two obtuſe angles, it has alſo one ſide greater than a quadrant.*

For by Prop. XVII, if two angles be greater than two right angles, then the ſum of their oppoſite ſides is greater than a ſemicircle; therefore at leaſt one of them is greater than a quadrant.

PROP. XXIV.

If the three angles of a ſpherical triangle be acute; each ſide will be leſs than a quadrant. 36.

For the perpendicular CD from any angle will fall within the triangle by Prop. XIX; and ſince, in the right-angled triangles ACD, and DCB, the angles A and ACD, and B and BCD are all acute, therefore by Cor. 1 Prop. XXI, the hypothenuſes AC, BC are leſs than a quadrant. And by the ſame reaſoning AB is leſs than a quadrant.

Cor. 1. *If the three ſides of a ſpherical triangle* DEF *be greater than quadrants, the three angles will be obtuſe.* 30.

For then the three angles A, C, B of the ſupplemental triangle ACB, will be acute, and therefore the ſides leſs than quadrants, by this Prop. And therefore by Prop. X, the angles of the triangle DEF will be obtuſe.

Cor.

FIG. 30. Cor. 2. *In a triangle* DEF, *if two angles* D, E *be obtuſe, and one* F *acute; the ſides are of the ſame affection as the oppoſite angles.*

For in the adjoining triangle DET all the angles are acute by Cor. 2 Prop. IX; and all the ſides leſs than quadrants, by this prop. and therefore in the triangle DEF; DF, EF are greater than quadrants.

Cor. 3. *In a triangle, if two ſides be leſſer, and one greater than a quadrant; the angles will be of the ſame affection as the oppoſite ſides.* This follows from Cor. 2 of this, and Prop. X.

PROP. XXV.

39. *In a ſpherical triangle, when two ſides are of the ſame affection, and the included angle acute; then the third ſide is leſs than a quadrant. But if the two ſides are of different affection, and the included angle obtuſe; the third ſide is greater than a quadrant.*

1. In the triangle BDA or BCA, right-angled at D and C, if the legs are of the ſame affection, the hypothenuſe BA is leſs than a quadrant, (by Prop. XXI) but if the angle at D or C be leſs than 90, as ACI, ADI; then AI is ſtill leſs.

2. Suppoſe the legs to be of different affection, then by (Prop. XXI) the hyp. BA will be greater than a quadrant, and much more will BA be greater than a quadrant, if the angle at C and D be obtuſe.

Cor. *When two angles are of the ſame affection, and the included ſide more than a quadrant; then the third angle is obtuſe. But if two angles are of different affection, and the included ſide leſs than a quadrant; the third angle is acute.*

For by Prop. X, its ſupplemental triangle comes under the caſe of this prop.

SECT. III.

The proportions for calculating the ſides and angles of ſpherical triangles; with ſome other things relating thereto.

PROP. XXVI.

In any right-angled ſpherical triangle ABC, *it is,* FIG. 40.
As radius :
To ſine of the hypothenuſe BC : :
So ſine of any angle C :
To ſine of its oppoſite ſide AB.

Let the triangle BAC be right-angled at A; draw AG, CG to the center G of the ſphere. From B let fall BI $\perp$ to AG, and thro' I draw IH $\perp$ to GC; and draw BH, which will be in the plane of the circle BC. Since the plane ABG is $\perp$ ACG, therefore BI is $\perp$ ACG, and CH $\perp$ to the plane BIH. Then BI is the ſine of BA, and BH the ſine of BC, and by Cor. 4 Prop. IV, $\angle$ BHI = ſpherical angle ACB.

This done, in the right-angled plain triangle BIH, it will be (by Prop. III B. II) rad. : BH : : ſin. H : BI; that is, rad. : ſin. BC : : ſin. C : ſin. AB.

Cor. 1. *In a right-angled triangle* ABF, *2 coſ. hyp.* AF × *rad.* = *coſ. ſum of the ſides* AB, BF + *coſ. their difference.* For in the triangle EDF, rad. : ſin. EF : : ſin. E : ſin. DF, that is, rad. : coſ. BF : : coſ. AB : coſ. AF, whence rad. × coſ. AF = coſ. AB × coſ. BF = (by Sch. 4, 1) coſ. $\overline{\text{AB} + \text{BF}}$ + coſ. AB — BF : × into ½ radius. 41.

PROP.

PROP. XXVII.

FIG. 40. *In any right-angled triangle* ABC, *as*
Radius :
Sine of one ſide AC : :
Tan. angle adjacent C :
Tan. oppoſite ſide AB.

Suppoſing the ſame conſtruction as in the laſt prop. then in the right-angled plane triangle HIG, it will be (by Prop. III B. II) rad. : ſin. AGC or AC : : IG : IH. And in the right-angled triangle BIG (right-angled at I) rad. : tan. BGI or AB : : IG : IB. And in the right-angled plane triangle BIH, tan. BHI or C : rad. : : IB : IH. Therefore, *ex equo*, tan. C : tan. AB : : IG : IH; that is, rad. : ſin. AC : : tan. C : tan. AB.

Cor. I.

91. In right-angled ſpherical triangles ABC, ADF, that have equal angles at the baſe; then

1. *The ſines of the hypothenuſes, are as the ſines of the perpendiculars.*
2. *The ſines of the baſes, are as the tangents of the perpendiculars.*
3. *The tangents of the hypothenuſes, are as the tangents of the baſes.*
4. *The coſines of the hypothenuſes, are as the cotangents of the angles at the vertex.*
5. *The coſines of the baſes, are as the coſines of the vertical angles.*
6. *The coſines of the perpendiculars, are reciprocally as the ſines of the angles at the vertex.*

All theſe are eaſily demonſtrated by the help of the two laſt propoſitions; by extending the ſides of the triangles to quadrants, after the manner of fig. 41; and

FIG. 91.

and finding the proportions, in the complemental triangles EDF, ADC, ECB; which will come out as follows:

As rad : S.A :: S.AC : S.CB :: S.AF : S.FD.
And rad : tan. A :: S.AB : tan. BC :: S.AD : tan. DF.
And rad : coſ. A :: tan. AC : tan. AB :: tan. AF : tan. AD.
And rad : tan. A :: coſ. AC : cotan. C :: coſ. AF : cotan. F.
And rad : S.A :: coſ. AB : coſ. C :: coſ. AD : coſ. F.
And rad × coſ. A = coſ. CB × S.C = coſ. FD × S.F.

Cor. 2.

In right-angled ſpherical triangles, AFB, AFD, that have equal hypothenuſes; then

1. *The ſines of the perpendiculars, are as the ſines of the angles at the baſe.*
2. *The ſines of the baſes are as the ſines of the vertical angles.*
3. *The tangents of the baſes, are as the coſines of the angles at the baſe.*
4. *The tangents of the angles at the baſe, are as the cotangents of the angles at the vertex.*
5. *The coſines of the baſes, are reciprocally as the coſines of the perpendiculars.*
6. *The tangents of the perpendiculars, are as the coſines of the vertical angles.*

For rad : S.AF :: S.FAB : S.FB :: S.FAD : S.FD.
And rad : S.AF :: S AFB : S.AB :: S.AFD : S.AD.
And rad : tan. AF :: coſ. FAB : tan. AB :: coſ. FAD : tan. AD.
And rad : coſ. AF :: tan. FAB : cotan. AFB :: tan. FAD : cotan. AFD.
And rad × coſ. AF = coſ. AB × coſ. BF = coſ. AD × coſ. DF.
And rad : tan. AF :: coſ. AFB : tan. FB :: coſ. AFD : tan. FD.

Cor. 3.

In right-angled ſpherical triangles ABC, ABF, having equal baſes; then

FIG. 91. 1. *The tangents of the perpendiculars, are as the tangents of the angles at the baſe.*

2. *The ſines of the angles at the baſe, are as the coſines of the angles at the vertex.*

3. *The ſines of the perpendiculars, are as the cotangents of the vertical angles.*

4. *The coſines of the perpendiculars, are as the coſines of the hypothenuſes.*

5. *The cotangents of the hypothenuſes, are as the coſines of the angles at the baſe.*

6. *The ſines of the hypothenuſes, are reciprocally as the ſines of the vertical angles.*

$$\text{For rad} : \text{s.}AB :: \begin{cases} \text{tan.}\,CAB : \text{tan.}\,CB. \\ \text{tan.}\,FAB ; \text{tan.}\,FB. \end{cases}$$

$$\text{And rad} : \text{coſ.}\,AB :: \begin{cases} \text{s.}\,CAB : \text{coſ.}\,ACB. \\ \text{s.}\,FAB : \text{coſ.}\,F. \end{cases}$$

$$\text{And rad} : \text{tan.}\,AB :: \begin{cases} \text{cotan.}\,ACB : \text{s.}CB. \\ \text{cotan.}\,F : \text{s.}FB. \end{cases}$$

$$\text{And rad} : \text{coſ.}\,AB :: \begin{cases} \text{coſ.}\,CB : \text{coſ.}\,AC. \\ \text{coſ.}\,FB : \text{coſ.}\,AF. \end{cases}$$

$$\text{And rad} ; \text{cotan.}\,AB :: \begin{cases} \text{coſ.}\,CAB : \text{cotan.}\,AC. \\ \text{coſ.}\,FAB : \text{cotan.}\,AF. \end{cases}$$

$$\text{And rad} \times \text{s.}AB = \text{s.}ACB \times \text{s.}AC \text{ or } = \text{s.}F \times \text{s.}AF.$$

Note, Either ſide may be taken for the baſe.

PROP. XXVIII.

41. *In any right-angled or quadrantal triangle; the ſine of middle part and radius, are reciprocally proportional to the tangents of the extremes conjunct, and to the coſines of the extremes disjunct.*

This is the Lord *Neper*'s theorem, and is demonſtrated by induction thus:

Produce the ſides AB, AF, BF to quadrants, and deſcribe the arch CDE. Then ſince the arches AC,

AD,

FIG. 41.

AD, BE, CE are quadrants, the angles B, C, D will be right; and ED = complement of DC = comp. A; and FD = comp. AF; and EF = comp. FB. Alſo the coſine of the complement is the ſine, and the cotan. of the comp. is the tangent.

Now the middle part muſt be either a leg, the comp. of an angle, or the comp. of the hypothenuſe.

1. Suppoſe a leg as AB be middle part, then BF and comp. ∠ A are extremes conjunct. But by Prop. XXVII, rad. : ſin. AB :: tan. A : tan. FB. Or thus, ſin. AB : tan. FB :: (rad. : tan. A ::) cotan. A : radius, by Cor. 2 Prop. 1 Book 1.

2. Let comp. of an angle A be middle part, then AB and comp. AF are extremes conjunct; and in the triangle DEF, by Prop. XXVII, rad. : ſin. ED :: tan. E : tan. DF; or ſin. ED : tan. DF :: (rad. : tan. E ::) cotan. E : radius, by Cor. 2 Prop. 1. 1; that is, coſ. A : cotan. AF :: tan. AB : rad.

3. Let the comp. hypothenuſe AF be middle part, then comp. A and comp. F are extremes conjunct. Therefore in the triangle DEF, rad. : ſin. DF :: tan. F : tan. ED; or ſin. DF : tan. ED (:: rad. : tan. F.) :: cotan. F : radius; that is, coſ. AF : cotan. A :: cotan. F : radius.

Therefore radius × ſin. middle part = rectangle of the tangents of the extremes conjunct. Again,

1. Let a leg AB be middle part, then comp. F, and comp. AF are extremes disjunct, therefore by Prop. XXVI, rad. : ſin. AF :: ſin. F : ſin. AB.

2. Let comp. of an angle A be middle part; then FB and comp. F are extremes disjunct: therefore in the triangle DEF, by Prop. XXVI, rad. : ſin. EF :: ſin. F : ſin. ED; that is, rad. : coſ. BF :: ſin. F : coſ. A.

3. Let comp. hyp. AF be middle part; then AB, FB are extremes disjunct. And in the triangle DEF,

FIG. 41. by Prop. XXVI, rad. : ſin. EF : : ſin. E : ſin. DF; that is, rad. : coſ. BF : : coſ. AB : coſ. AF.

Therefore radius × ſine of middle part = rectangle of the coſines of the extremes disjunct.

And in a quadrantal triangle, its ſupplemental one, by Prop. X, will be a right-angled triangle. And ſince this prop. holds in that right-angled triangle, it will equally hold in the quadrantal triangle, wherein the parts are the ſupplements of the others; becauſe arches and their ſupplements have the ſame ſines, coſines, tangents, and cotangents.

42. Cor. 1. *If a perpendicular* FI *be let fall on the baſe of a ſpherical triangle, the coſines of the angles at the baſe are as the ſines of the angles at the vertex.*

For in the right-angled triangles AFI, BFI, it is coſ. A : ſin. AFI : : (coſ. FI : rad. : :) coſ. FBI : ſin. BFI.

Cor. 2. *The coſines of the ſides are as the coſines of the ſegments of the baſe.*

For coſ. AI : coſ. AF : : (rad. : coſ. FI : :) coſ. BI : coſ. BF.

Cor. 3. *The ſines of the ſegments of the baſe are as the cotangents of the angles at the baſe.*

For cotan. A : ſin. AI : : (rad. : tan. FI : :) cot. IBF : ſin. IB.

Cor. 4. *The cotangents of the ſides, are as the coſines of the angles at the vertex.*

For cotan. AF : coſ. AFI : : (rad. : tan. FI : :) cotan. BF : coſ. BFI.

Cor. 5. *The tangents of the ſegments of the baſe are as the tangents of the angles at the vertex.*

For tan. AFI : tan. AI : : (rad. : ſin. FI : :) tan. BFI : tan. BI.

Cor. 6. *The ſines of the ſides, are reciprocally as the ſines of the angles at the baſe.* FIG. 42.

$$\text{For ſin. } AF : \text{rad} :: \text{ſin. } FI : \text{ſin. } FAI = \frac{r \times \text{ſin. } FI}{\text{ſin. } AF},$$

$$\text{and ſin. } BF : \text{rad} :: \text{ſin. } FI : \text{ſin. } FBI = \frac{r \times \text{ſin. } FI}{\text{ſin. } BF};$$

$$\text{whence ſin. } FAI : \text{ſin. } FBI :: \frac{1}{\text{ſin. } AF} : \frac{1}{\text{ſin. } BF}.$$

SCHOLIUM.

This prop. will reſolve all right-angled and quadrantal triangles; and alſo all oblique ones, (except where 3 ∠s or 3 ſides are given,) by letting fall a perpendicular dividing it into two right-angled triangles.

PROP. XXIX.

In any ſpherical triangle, the ſines of the ſides are proportional to the ſines of their oppoſite angles. 43.

On AB, AF let fall the perpendiculars FH, BI; then by Prop. XXVI.

ſin. AF : ſin. FH :: *rad.* : ſin. A
and ſin. FH : ſin. FB :: ſin. B : *rad.* therefore,
ex equo, ſin. AF : ſin. FB :: ſin. B : ſin. A.

again ſin. AB : ſin. BI :: *rad.* : ſin. A
and ſin. BI : ſin. BF :: ſin. F : *rad.* and,
ex equo, ſin. AB : ſin. BF :: ſin. F : ſin. A.

Cor. 1. *If a great circle be drawn from the vertex of a triangle to cut the baſe; the rectangles of the ſines, of the ſides and of the vertical angles; are directly as the ſines of the ſegments of the baſe.* 44.

FIG. 44. For ſin. CDA : ſin. CA :: ſin. CAD : ſin. CD = $\frac{\text{ſin. CA} \times \text{ſin. CAD}}{\text{ſin. CDA}}$, and ſin. CDA or BDA : ſin. AB :: ſin. DAB : ſin. DB = $\frac{\text{ſin. AB} \times \text{ſin. DAB}}{\text{ſin. CDA}}$; therefore ſin. CD : ſin. DB :: ſin. CA × ſin. CAD : ſin. BA × ſin. DAB.

Cor. 2. *If the vertical angle of a triangle be biſected; the ſines of the ſegments of the baſe are as the ſines of the ſides;* ſin. CD : ſin. BD :: ſin. AC : ſin. AB.

PROP. XXX.

42. *In a ſpherical triangle, if a perpendicular be let fall on the baſe, it is*

As the ſine of the ſum of the ſides, AF, BF :
Sine of their difference ::
Cot. of ½ *ſum of the angles at the vertex,* AFI, BFI :
Tan. of half their difference.

For by Cor. 4 Pr. XXVIII, cotan. AF : cotan. BF :: coſ. AFI : coſ. BFI; or by Cor. 4 Pr. I B. I, tan. AF : tan. BF :: coſ. BFI : coſ. AFI. And by compoſition and diviſion, tan. AF + tan. BF : tan. AF — tan. BF :: coſ. BFI + coſ. AFI : coſ. BFI — coſ. AFI; that is, by Cor. 4 and 2 Pr. VII B. I, ſin. $\overline{\text{AF} + \text{BF}}$: ſin. $\overline{\text{AF} - \text{BF}}$ :: cotan. $\frac{\text{AFI} + \text{BFI}}{2}$: tan. $\frac{\text{AFI} - \text{BFI}}{2}$.

Cor. 1. *If the perpendicular falls within,* $\frac{\text{AFI} + \text{BFI}}{2}$ = *half the angle at the vertex* F : *if it falls without,* $\frac{\text{AFI} - \text{BFI}}{2}$ = *half the vertical angle.*

FIG.

Cor. 2. *As ſine of the ſum of the ſides :*
Sine of difference of the ſides : : 42.
Cotan. half the vertical angle :
Tan. half difference, or half ſum, of the vertical angles, according as the perpendicular falls within or without.

Cor. 3. *In a right-angled ſpherical triangle* ABF, 52.
Sin. ſum of the hyp. and one ſide, AF, BF *:*
Sin. of their difference : :
Radius ſquare :
Tang. ſquare of half the contained angle F.

For in this caſe the ∠ F is both the ſum of the angles, and the difference of the angles. And cot. $\frac{1}{2}$F : tan. $\frac{1}{2}$F :: $\frac{\text{Rad.}^2}{\text{Tan. }\frac{1}{2}F}$: tan. $\frac{1}{2}$F :: rad.^{2} : tan.^{2} of $\frac{1}{2}$F.

PROP. XXXI.

Let a perpendicular fall on the baſe of a triangle; then 42.
Tan. half the ſum of the ſides, AF, BF *:*
Tan. half their difference : :
Tan. half the ſum of the angles at the baſe, B, A *:*
Tan. half their difference.

For by Prop. XXIX, ſin. AF : ſin. BF :: ſin. B : ſin. A. And by compoſition and diviſion, ſin. AF + ſin. BF : ſin. AF — ſin. BF :: ſin. B + ſin. A : ſin. B — ſin. A; that is, by Cor. 1 Pr. VII, B. I, tan. $\frac{AF + BF}{2}$: tan. $\frac{AF - BF}{2}$ :: tan. $\frac{B + A}{2}$: tan. $\frac{B - A}{2}$.

PROP. XXXII.

FIG. 42. *Let a perpendicular fall on the baſe of a ſpherical triangle; then*

Cotan. half the ſum of the angles at the baſe, A, B :
Tan. half their difference ::
Tan. ½ *ſum of the angles at the vertex,* AFI, BFI :
Tan. half their difference.

For by Cor. 1 Prop. XXVIII, coſ. A : coſ. B :: ſin. AFI : ſin. BFI. And coſ. A + coſ. B : coſ. A — coſ. B :: ſin. AFI + ſin. BFI : ſin. AFI — ſin. BFI; that is, by Cor. 1 and 2 Prop. 7 B. 1, cotan. $\frac{B+A}{2}$: tan. $\frac{B-A}{2}$ :: tan. $\frac{AFI+BFI}{2}$: tan. $\frac{AFI-BFI}{2}$.

Note, If the perpendicular falls within, $\frac{AFI+BFI}{2}$ = half the vertical angle F; but $\frac{AFI-BFI}{2}$, if it fall without.

Cor. *Cotan. half the ſum of the angles at the baſe* :
Tan. half their difference ::
Tan. half the vertical angle :
Tan. half the difference, or half the ſum of the vertical angles, as the perpendicular falls within or without.

This is plain when the perpendicular falls within; and when it falls without, + BFI becomes — BFI, and the contrary.

PROP. XXXIII.

Suppoſe a perpendicular to fall on the baſe of a ſpherical triangle, then as FIG. 42.

Tan. half ſum of the ſegments of the baſe AI, BI :
Tan. half the ſum of the ſides, AF, BF : :
Tan. half difference of the ſides AF, BF :
Tan. ½ *diff. of the ſegments of the baſe,* AI, BI.

For by Cor. 2 Prop. XXVIII, coſ. BF : coſ. AF : : coſ. BI : coſ. AI. And coſ. BF + coſ. AF : coſ. BF — coſ. AF : : coſ. BI + coſ. AI : coſ. BI — coſ. AI; that is, by Cor. 2 Prop. VII B. I, cotan. $\frac{AF + BF}{2}$: tan. $\frac{AF - BF}{2}$: : cotan. $\frac{AI + BI}{2}$: tan. $\frac{AI - BI}{2}$.

Or by Cor. 4 Prop. I B. I, tan. $\frac{AI + BI}{2}$: tan. $\frac{AF + BF}{2}$: : (cotan. $\frac{AF + BF}{2}$: cotan. $\frac{AI + BI}{2}$: :) tan. $\frac{AF - BF}{2}$: tan. $\frac{AI - BI}{2}$.

Here as the perpendicular falls within or without, AI + BI or AI — BI will be the baſe.

Cor. 1. *As tan. half the baſe* :
Tan. half ſum of the ſides : :
Tan. half difference of the ſides :
Tan. half the alternate baſe.

Where the alternate baſe is half the difference, or half the ſum of the ſegments of the baſe, according as the perpendicular falls within or without.

Cor. 2. *In a right-angled triangle; the rectangle of the tan. of half the hypothenuſe* + *half one leg, and the*

FIG. *the tan. half the hypothenuſe — half that leg = tan. ſquare of half the other leg.* For then BI = 0.

PROP. XXXIV.

42. *Let fall a perpendicular upon the baſe of a triangle, then*
Sine of the ſum of the angles at the baſe B, A :
Sine of their difference : :
Tan. half ſum of the ſegments of the baſe, AI, BI :
Tan. half their difference.

For by Cor. 3 Prop. XXVIII, cotan. A : cotan. B : : ſin. AI : ſin. BI, or tan. B : tan. A : : ſin. AI : ſin. BI. And tan. B + tan. A : tan. B — tan. A : : ſin. AI + ſin. BI : ſin. AI — ſin. BI ; that is, (by Cor. 4 and 1 Pr. VII B. I) ſin. B + A : ſin. B — A : : tan. $\frac{AI + BI}{2}$: tan. $\frac{AI - BI}{2}$.

According as the perpendicular falls within or without, $\frac{AI + BI}{2}$, or $\frac{AI - BI}{2}$ will be equal to half the baſe. Whence

Cor. 1. *Sine of the ſum of the angles at the baſe* :
Sine of their difference : :
Tan. half the baſe :
Tan. half the alternate baſe. (See Cor. 1 Pr. 33)

For when the perpendicular falls without, + BI becomes — BI, and the contrary.

PROP. XXXV.

Let fall a perpendicular on the baſe, then as
Sine of the ſum of the ſegments of the baſe, AI, BI :
Sine of their difference : :
Sine of the ſum of the angles at the vertex, AFI, BFI :
Sine of their difference.

FIG. 42.

For by Cor. 5 Prop. XXVIII, tan. AI : tan. BI :: tan. AFI : tan. BFI. And tan. AI + tan. BI : tan. AI — tan. BI :: tan. AFI + tan. BFI : tan. AFI — tan. BFI; that is, by Cor. 4 Pr. VII B. I, ſin. $\overline{AI+BI}$: ſin. $\overline{AI-BI}$ :: ſin. $\overline{AFI+BFI}$: ſin. $\overline{AFI-BFI}$. Here, as the perpendicular falls within or without; AI + BI, or AI — BI is the baſe; and AFI + BFI, or AFI — BFI, the vertical angle. Whence

Cor. *As ſine of the baſe :*
Sine of the vertical angle ::
So ſine of the diff. ſegments of the baſe :
Sine of the diff. angles at the vertex, when the perpendicular falls within ::
Or ſine of the ſum of the ſegments of the baſe :
Sine of the ſum of the vertical angles, when the perpendicular falls without.

PROP. XXXVI.

42.

As coſine of half the ſum of two ſides, AF, BF :
Coſine of half their difference ::
Cotan. half the included angle AFB :
Tan. half ſum of the oppoſite angles A, ABF.

Put AF + BF = P, AF — BF = p, ABF + A = Q, ABF — A = q, AI + BI = R, AI — BI = r, AFI + BFI = V, AFI — BFI = v.

Then ſince the tangents are reciprocally as the cotangents, we ſhall have, by

Prop. 30, ſin. P : ſin. p :: *cotan.* $\frac{1}{2}$ V : tan. $\frac{1}{2}$ v.
31, *Cot.* $\frac{1}{2}$ P : *cot.* $\frac{1}{2}$ p :: tan. $\frac{1}{2}$ q : tan. $\frac{1}{2}$ Q.
32, tan. $\frac{1}{2}$ q : tan. $\frac{1}{2}$ v :: *cot.* $\frac{1}{2}$ V : tan. $\frac{1}{2}$ Q.

And compounding theſe ratios, and expunging equal quantities,

Sin. P

FIG. 42. Sin. $\mathrm{P}\times$cot. $\frac{1}{2}\mathrm{P}$: ſin. $p\times$cot. $\frac{1}{2}p$: : cot. ſquare $\frac{1}{2}\mathrm{V}$: tan. ſquare $\frac{1}{2}\mathrm{Q}$. But by ſch. Pr. 2 B. I, $\frac{2\,\overline{\text{coſ.}}^{2}\,\frac{1}{2}\mathrm{P}}{\text{cot.}\,\frac{1}{2}\mathrm{P}}$ $=$ ſin. P, or $\overline{\text{coſ.}}^{2}\,\frac{1}{2}\mathrm{P} = \frac{\text{ſin.}\,\mathrm{P}\times\text{cot.}\,\frac{1}{2}\mathrm{P}}{2}$, and $\overline{\text{coſ.}}^{2}\,\frac{1}{2}p = \frac{\text{ſin.}\,p\times\text{cot.}\,\frac{1}{2}p}{2}$, therefore coſ. ſquare $\frac{1}{2}\mathrm{P}$: coſ. ſquare $\frac{1}{2}p$: : cot. ſquare $\frac{1}{2}\mathrm{V}$: tan. ſquare $\frac{1}{2}\mathrm{Q}$; and coſ. $\frac{1}{2}\mathrm{P}$: coſ. $\frac{1}{2}p$: : cot. $\frac{1}{2}\mathrm{V}$: tan. $\frac{1}{2}\mathrm{Q}$.

This is plain when the perpendicular falls within; and when it falls without, inſtead of the third proportion, ſubſtitute this (by Cor. Pr. XXXII) tan. $\frac{1}{2}q$: tan. $\frac{1}{2}\mathrm{V}$: : cot. $\frac{1}{2}v$: tan. $\frac{1}{2}\mathrm{Q}$, and you will have at laſt this; coſ. $\frac{1}{2}\mathrm{P}$: coſ. $\frac{1}{2}p$: : cot. $\frac{1}{2}v$: tan. $\frac{1}{2}\mathrm{Q}$.

Cor. 1. *As ſine of half the ſum of the ſides :*
Sine of half their difference : :
Cotan. half the included angle :
Tan. half difference of the oppoſite angles.

For by the former proportions, a little varied.

ſin. P : ſin. p : : cot. $\frac{1}{2}\mathrm{V}$: tan. $\frac{1}{2}v$.
tan. $\frac{1}{2}\mathrm{P}$: tan. $\frac{1}{2}p$: : tan. $\frac{1}{2}\mathrm{Q}$: tan. $\frac{1}{2}q$,
tan. $\frac{1}{2}\mathrm{Q}$: tan. $\frac{1}{2}v$: : cot. $\frac{1}{2}\mathrm{V}$: tan. $\frac{1}{2}q$.

Then proceeding as before, it will be found (by Sch. 2 Prop. 2 B. I) that

Sin. $\frac{1}{2}\mathrm{P}$: ſin. $\frac{1}{2}p$: : cot. $\frac{1}{2}\mathrm{V}$: tan. $\frac{1}{2}q$, when the perpendicular falls within; and when it falls without, inſtead of the third proportion put this (by Cor. Prop. XXXII) cot. $\frac{1}{2}\mathrm{V}$: cot. $\frac{1}{2}v$: : cot. $\frac{1}{2}\mathrm{Q}$: tan. $\frac{1}{2}q$, and you will get at laſt ſin. $\frac{1}{2}\mathrm{P}$: ſin. $\frac{1}{2}p$: : cot. $\frac{1}{2}v$: tan. $\frac{1}{2}q$.

Cor. 2. *Coſ. half ſum of two ſides :*
Coſ. half their difference : :

Tan. half difference of their oppoſite angles :
Tan. half difference, or half ſum of the vertical angles; *according as the perpendicular, on the third ſide, falls within or without.*

This follows from the preſent prop. and cor. Prop. XXXII.

PROP. XXXVII.

42.

As coſ. half ſum of two angles, A, ABF :
Coſ. half their difference : :
Tan. half the included ſide AB :
Tan. half ſum of the oppoſite ſides, AF, BF.

Suppoſing as in the laſt prop. then by

Prop. 31, cot. $\frac{1}{2}Q$: cot. $\frac{1}{2}q$: : tan. $\frac{1}{2}p$: tan. $\frac{1}{2}P$,
33, tan. $\frac{1}{2}p$: tan. $\frac{1}{2}r$: : tan. $\frac{1}{2}R$: tan. $\frac{1}{2}P$;
34, ſin. Q : ſin. q : : tan. $\frac{1}{2}R$: tan. $\frac{1}{2}r$.

But (by Sch. Prop. II B. I) $\frac{1}{2}$ ſin. $Q \times$ cot. $\frac{1}{2}Q = \overline{\text{coſ.}}^2\,\frac{1}{2}Q$, and $\frac{1}{2}$ ſin. $q \times$ cot. $\frac{1}{2}q = \overline{\text{coſ.}}^2\,\frac{1}{2}q$, therefore by compounding,

$\overline{\text{Coſ.}}^2\,\frac{1}{2}Q : \overline{\text{coſ.}}^2\,\frac{1}{2}q :: \overline{\text{tan.}}^2\,\frac{1}{2}R : \overline{\text{tan.}}^2\,\frac{1}{2}P.$
Or coſ. $\frac{1}{2}Q$: coſ. $\frac{1}{2}q$: : tan. $\frac{1}{2}R$: tan. $\frac{1}{2}P$.

But when the perdendicular falls without, inſtead of the third proportion, put this (by Cor. Prop. XXXIV) ſin. Q : ſin. q : : tan. $\frac{1}{2}r$: tan. $\frac{1}{2}R$. Whence at laſt you will have this,

Coſ. $\frac{1}{2}Q$: coſ. $\frac{1}{2}q$: : tan. $\frac{1}{2}r$: tan. $\frac{1}{2}P$.

Cor. I. *Sine half ſum of two angles*, A, ABF :
Sine of half their difference : :
Tan. half the included ſide, AB :
Tan. half difference of the oppoſite ſides, AF, BF.

For by varying the terms of the foregoing proportions, it will be,

tan.

FIG. 42. tan. $\frac{1}{2}$Q : tan. $\frac{1}{2}q$ :: tan. $\frac{1}{2}$P : tan. $\frac{1}{2}p$,
tan. $\frac{1}{2}$P : tan. $\frac{1}{2}r$ :: tan. $\frac{1}{2}$R : tan. $\frac{1}{2}p$,
ſin. Q : ſin. q :: tan. $\frac{1}{2}$R : tan. $\frac{1}{2}r$.

But (by Sch. Prop. II Book I) $\frac{1}{2}$ſin. Q × tan. $\frac{1}{2}$Q = ſin.^{2} $\frac{1}{2}$Q, and $\frac{1}{2}$ſin. q × tan. $\frac{1}{2}q$ = ſin.^{2} $\frac{1}{2}q$, therefore by compounding, ſin. $\frac{1}{2}$Q : ſin. $\frac{1}{2}q$ :: tan. $\frac{1}{2}$R : tan. $\frac{1}{2}p$.

If the perpendicular fall without, inſtead of the third proportion put this (by Cor. Prop. XXXIV) ſin. Q : ſin. q :: tan. $\frac{1}{2}r$: tan. $\frac{1}{2}$R, which will give this proportion, ſin. $\frac{1}{2}$Q : ſin. $\frac{1}{2}q$ :: tan. $\frac{1}{2}r$: tan. $\frac{1}{2}p$.

Cor. 2. *Coſ. $\frac{1}{2}$ ſum of the angles at the baſe,* A, ABF :
Coſ. half their difference ::
Tan. half diff. oppoſite ſides, AF, BF :
Tan. half diff. or half ſum, of the ſegments of the baſe; according as the perpendicular falls within or without.

This follows from the preſent prop. and Cor. I Prop. XXXIII.

PROP. XXXVIII.

42. *In any ſpherical triangle* AFB; *the rectangle of the ſines of any two ſides × coſ. included angle + the rectangle of their coſines × radius = coſ. of the third ſide × radius ſquare; that is,*

ſin. AB × ſin. FB × *coſ.* B + *coſ.* AB × *coſ.* FB × *rad.* = *coſ.* AF × *rad*^{2}.

Let a = ſin. AB, m = coſ. AB, s = ſin. B.
b = ſin. FB, n = coſ. FB, c = coſ. B, t = cot. B.
r = rad. $p = \sqrt{rrnn + bbcc}$.

By Prop. I. I, $n : b :: r$: tan. FB :: cot. FB : r :: (by Prop. XXVIII) c : tan. BI $= \frac{cb}{n}$. And by Sch. Prop. I. I, ſine of BI $= \frac{rcb}{p}$, and coſ. BI $= \frac{rrn}{p}$;

$\frac{rrn}{p}$; and by Cor. Pr. 6. I, coſ. AI $= \frac{abc + mnr}{p}$, FIG. 42.
but by Cor. 2 Prop. XXVIII, $\frac{rrn}{p} : \frac{abc + mnr}{p} ::$ $n :$ coſ. AF $= \frac{abc + mnr}{rr}$.

Cor. I. *The coſine of any angle of a ſpherical triangle is equal to the coſine of the oppoſite ſide × radius ſquare — the rectangle of the coſines of the including ſides × radius, and the whole divided by the rectangle of the ſines of the including ſides.*

For coſ. B $= \frac{\text{coſ. AF} \times rr - \text{coſ. AB} \times \text{coſ. FB} \times r}{\text{ſin. AB} \times \text{ſin. FB}}$

Cor. 2. *Cotan.* A $=$
$$\frac{\text{ſin. AB} \times \textit{coſ.}\ \text{FB} \times \textit{rad.} - \text{ſin. FB} \times \textit{coſ.}\ \text{AB} \times \textit{coſ.}\ \text{B}}{\text{ſin. B} \times \text{ſin. FB}}$$

For by Prop. 6. I, ſin. AI $= \frac{anr - bcm}{p}$, and by Cor. 3 Prop. XXVIII. I, $\frac{rcb}{p} : \frac{anr - bcm}{p} :: r :$ cot. A $=$ $\frac{anr - bcm}{rcb} r = \frac{anr - mbc}{bs}$, becauſe $\frac{r}{c} = \frac{r}{s}$ by Schol. Prop. I. I.

Cor. 3, $\frac{\text{ſ.}\,\overline{\text{AB} - \text{FB}} \times rr + \text{ſ. FB} \times \textit{coſ.}\ \text{AB} \times \textit{verſ.}\ \text{B}}{\text{ſin. B} \times \text{ſin. FB}}$ $=$ *cotan.* A.

For ſince coſ. B $= r -$ verſ. B, therefore by Cor. 2, cot. A $= \frac{anr - mbr + mb \times \text{verſ. B}}{bs}$; but by Pr. 6. I, $an - bm =$ ſin. $\overline{\text{AB} - \text{FB}} \times r$, whence the Cor. is evident.

SCHO-

SCHOLIUM.

FIG. 42. If the angle ABF, or either ſide, be greater than a right angle, then its coſine c will be negative.

PROP. XXXIX.

45. *The rectangles of the ſines of two angles × coſine of the included ſide divided by radius ſquare — the rectangle of their coſines divided by radius, is equal to the coſine of the third angle.*

that is, $\frac{\text{ſin. A} \times \text{ſin. B} \times \textit{coſ.}\ \text{AB}}{rr} - \frac{\textit{coſ.}\ \text{A} \times \textit{coſ.}\ \text{B}}{r} = \textit{coſ.}\ \text{F}$.

Let fall the perpendicular BR, and put
ſin. A $= a$, coſ. A $= m$, cot. A $= \tau$, coſ. AB $= c$,
ſin. B $= b$, coſ. B $= n$, $q = \sqrt{\tau\tau + cc}$, rad. $= r$.

Then by Prop. XXVIII, cot. ABR $= \frac{rc}{\tau}$, and (by Sch. 1. 1) its ſine $= \frac{r\tau}{q}$, and its coſine $= \frac{rc}{q}$. And by Prop. 6. 1, ſin. RBF $= \frac{bc - n\tau}{q}$; and by Cor. 1. XXVIII, $\frac{r\tau}{q} : \frac{bc - n\tau}{q} :: m :$ coſ. F $= \frac{bc - n\tau}{r\tau} m = \frac{bcm}{r\tau} - \frac{mn}{r} = \frac{abc}{rr} - \frac{mn}{r}$; becauſe $\frac{m}{\tau} = \frac{a}{r}$, by Sch. 1. 1.

Cor. 1. *The coſine of any ſide is equal to the rectangle of the coſines of the including angles × radius + coſ. oppoſite angle × radius ſquare, and the whole divided by the rectangles of the ſines of the including angles.*

For c or the *coſ.* AB $= \frac{mnr + rr \times \text{coſ. F}}{ab}$.

Cor. 2. $\text{Cot. FB} = \frac{\text{ſ. B} \times \text{cot. A}}{\text{ſ. AB}} + \frac{\text{coſ. B} \times \text{cot. AB}}{r}$. FIG. 45.

For let $s = \text{ſin. AB}$, $t = \text{cotan. AB}$, then by Cor. 1 Prop. 6. 1, $\text{coſ. RBF} = \frac{b\tau + nc}{q}$, and by Cor. 4 Prop. XXVIII, $\frac{rc}{q} : \frac{b\tau + nc}{q} :: t : \text{cot. BF} = \frac{b\tau + nc}{rc} t = \frac{b\tau}{s} + \frac{nt}{r}$, becauſe $ts = cr$. Or $\text{cot. BF} = \frac{rbm + nac}{sa}$, becauſe $\tau = \frac{mr}{a}$, and $t = \frac{cr}{s}$, by Schol. Prop. 1. 1.

Cor. 3. $\text{Cotan. FB} =$

$$\frac{r^2 \times \text{ſ.}\,\overline{\text{A} + \text{B}} - \text{ſ. A} \times \text{coſ. B} \times \text{verſ. AB}}{\text{ſin. A} \times \text{ſin. AB}}.$$

For let $v = \text{verſ. AB} = r - c$, and $c = r - v$, then $\text{cotan. FB} = \frac{rbm + nac}{sa} = \frac{rbm + ran - anv}{sa}$; but (by Prop. 5. 1) $mb + an = r \times \text{ſin.}\,\overline{\text{A} + \text{B}}$, whence $\text{cot. BF} = \frac{rr \times \text{ſin. A} + \text{B} - anv}{as}$.

PROP. XL.

Let a, s, b, p, f, d *be the ſines,* *and* m, c, n, i, g, e *the coſines* *of the angles and ſides as mark'd in the figure. And* 47.
$hh = bb - aa = mm - nn$.
and $kk = 1 - bbpp = 1 - aadd$.
Radius $= 1$. *Then*

Sine of $\angle\,\text{F} = \frac{hh}{bim - aen} = \frac{bim + aen}{kk}$.

FIG. 47. For 1. $bif - ng = m$, by Prop. XXXIX,
2. $aef - mg = n$, by Prop. XXXIX,
3. $\frac{n}{b} = \frac{pe - dig}{df}$. By Cor. 2 Pr. XXXVIII.

For by Pr. 1, 1, $\frac{b}{n} =$ cotan. B. From the two firſt equations $g = \frac{bif - m}{n} = \frac{aef - n}{m}$, whence $bifm - m^2 = aefn - nn$, and $f = \frac{m^2 - n^2}{bim - aen}$.

Again, by the third equation, $dnf = bpe - bdig = ade - bdig$, becauſe (by Prop. XXIX) $bp = ad$; and $nf = ae - big$; whence $g = \frac{ae - nf}{bi} = \frac{bif - m}{n}$, and $aen - fn^2 = b^2i^2f - bim$, whence
$f = \frac{aen + bim}{n^2 + bbii} = \frac{aen + bim}{1 - bbpp}$; becauſe $ii = 1 - pp$.

Cor. 1. Coſ. F $= \frac{abei - mn}{kk} = \frac{aem - bin}{bim - aen}$.

For, by the 1ſt and 3d equations, $f = \frac{ng + m}{bi} = \frac{ae - big}{n}$, and $nng + mn = aebi - bbiig$; whence
$g = \frac{aebi - mn}{nn + bbii} = \frac{aebi - mn}{1 - bbpp}$.
Again, by the 1ſt and 2d equations,
$f = \frac{m + ng}{bi} = \frac{n + mg}{ae}$, and $aem + aeng = bin + bimg$,
whence $g = \frac{aem - bin}{bim - aen}$.

Cor.

FIG. 47.

Cor. 2. *Cotan.* F $= \frac{ame - bin}{bb} = \frac{e \times \text{ſ}.2\text{A} - i \times \text{ſ}.2\text{B}}{co\text{ſ}.2\text{A} - co\text{ſ}.2\text{B}}$.

For multiplying the 1ſt equation by n, and the 2d by m, we have $mn = binf - nng = aefm - mmg$, and $\overline{m^2 - n^2}.g = \overline{aem - bin}.f$, and $\frac{g}{f} = \frac{aem - bin}{m^2 - n^2} = \frac{2aem - 2bin}{2b^2 - 2a^2}$; but (by Prop. 2. I) $2am =$ ſin. 2A, and $2bn =$ ſin. 2B, alſo $2aa =$ verſ. 2A, and $2b^2 =$ verſ. 2B, and $2bb - 2aa =$ verſ. 2B — verſ. 2A $=$ coſ. 2A — coſ. 2B, and $b^2 - a^2 = m^2 - n^2$, for $bb + n^2 = 1 = m^2 + a^2$.

SCHOLIUM.

Any of the quantities a, b, p, d, or m, n, i, e, may be expunged by help of the equation $bp = ad$, or the tangents or cotangents ſubſtituted for them.

PROP. XLI.

47.

Suppoſe as in the laſt Prop. and let
$ll = dd - pp = ii - ee$,
$tt = 1 - bbpp = 1 - aadd$,
Radius $= 1$. *Then*
Sine of AB $= \frac{ll}{mid - pen} = \frac{mid + pen}{tt}$.

For 1. $pns + ci = e$, by Prop. XXXVIII,
2. $mds + ce = i$, by Prop. XXXVIII,
3. $\frac{i}{p} = \frac{bm + anc}{as}$, by Cor. 2 Pr. XXXIX.

By the two firſt equations, $c = \frac{e - pns}{i} = \frac{i - mds}{e}$, and $ee - pens = ii - imds$, and $imds - pens = ii - ee$, whence $s = \frac{ii - ee}{dim - pen}$.

Alſo

FIG. 47. Alſo by the third equation $asi = pbm + pnac =$ (ſince $ad = bp$) $adm + pnac$, that is, $si = dm + pnc$, whence $c = \frac{si - dm}{pn} = \frac{e - pns}{i}$, and $si^2 - dim =$ $pen - p^2n^2s$, whence $s = \frac{pen + dim}{i^2 + p^2n^2} = \frac{pen + dim}{1 - ppbb}$.

Cor. 1. *Coſ.* AB $= \frac{ei - mndp}{tt} = \frac{pin - dem}{pen - dim}$.

For by 1st and 3d equations, $s = \frac{e - ci}{pn} = \frac{dm + cpn}{i}$, and $ei - ci^2 - pndm + p^2n^2c$, whence $c = \frac{ei - pdmn}{i^2 + p^2n^2} = \frac{ei - pdmn}{1 - bbpp}$.

Alſo, by the 1st and 2d equations, $s = \frac{e - ci}{pn} = \frac{i - ce}{md}$, and $mde - mdci = pni - pnec$, whence $c = \frac{dem - pin}{dim - pen}$.

47. Cor. 2. *Cotan.* AB $=$ $\frac{med - pin}{ll} = \frac{m \times \text{ſin. 2AF} - n \times \text{ſin. 2BF}}{\text{coſ. 2BF} - \text{coſ. 2AF}}$.

For multiplying the 1st equation by i, and the 2d by e, we have $pnsi + cii = ei = mdse + cee$, and $cii - cee = mdes - pins$, whence $\frac{c}{s} = \frac{med - pin}{ii - ee} = \frac{2med - 2pin}{2dd - 2pp}$; but by Prop. 2, 1, $2ed =$ ſin. 2AF, and $2pi =$ ſin. 2BF, and $2dd =$ verſ. 2AF, and $2p^2 =$ verſ. 2BF, and $2dd - 2pp =$ verſ. 2AF — verſ. 2BF = coſ. 2BF — coſ. 2AF.

FIG. 47.

SCHOLIUM.

Any of the quantities a, d, b, p; or m, e, i, n, may be exterminated by help of the equation $ad = bp$, or the tangents or cotangents ſubſtituted inſtead of them.

PROP. XLII.

In any ſpherical triangle, ABF, 46.
As rectangle of the ſines of the ſides, AB, FB :
Radius ſquare : :
Verſ. baſe — verſ. difference of the ſides :
Verſ. included angle B : :
Coſ. difference of the ſides — coſ. baſe :
Verſ. included angle B : :
As ſine of one ſide :
To coſecant of the other.

Let $a = \text{ſ. AB}$, $m = \text{coſ. AB}$, $c = \text{coſ. B}$, $v = \text{verſ. B}$.
$b = \text{ſ. FB}$, $n = \text{coſ. FB}$, $d = \text{coſ. AF}$, rad. $= r$.

Then (by Cor. 2 Prop. XXXVIII) $\frac{dr^2 - mnr}{ab} = c = r - v$, and $v = \frac{abr - drr + mnr}{ab}$; but (by Cor. 1 Prop. 6, 1.) $ab + \underline{mn = r} \times \text{coſ.}\ \overline{\text{AB} - \text{FB}}$. Therefore $v = \frac{rr \times \text{coſ.}\ \overline{\text{AB} - \text{FB}} - drr}{ab} = \frac{\text{verſ. AF} - \text{verſ.}\ \overline{\text{AB} - \text{FB}}}{ab} \times rr$. Laſtly, $\text{ſ. FB} = \frac{rr}{\text{coſec. FB}}$, and $\text{ſ. AB} \times \text{ſ. FB} : rr :: \text{ſ. AB} \times \frac{rr}{\text{coſec. FB}} : rr :: \text{S. AB} : \text{coſec. FB}$.

Cor. 1. *Rectangle of the ſines of the ſides* :
ſ. half ſum × ſ. half diff. of the baſe and diff. ſides : :
And verſ. ſum of the ſides — verſ. their difference :
Verſ. baſe — verſ. difference of the ſides : :

FIG. 46. *And coſ. difference — coſ. ſum of the ſides :*
Coſ. diff. ſides — coſ. baſe : :
So twice radius :
Verſ. included angle, B.

For let the greater ſide AB $= a$, leſſer BF $= e$, baſe AF $= b$, $a + e = z$, $a - e = d$, $v =$ verſ. B.

Then by this Prop. ſ. $a \times$ ſ. $e : \overline{\text{verſ.}\, b - \text{verſ.}\, d} \times r :: r : v$. But (by Cor. 3 Pr. 3, I) $\frac{1}{2} r \times \overline{\text{verſ.}\, b - \text{verſ.}\, d} = \text{ſin.}\, \frac{b+d}{2} \times \text{ſin.}\, \frac{b-d}{2}$, and (by Cor. 2 Prop. 3, I) $\text{ſin.}\, a \times \text{ſin.}\, e = \frac{1}{2} r \times \overline{\text{coſ.}\, d - \text{coſ.}\, z}$. Whence you have theſe two proportions,

$$\text{ſin.}\, a \times \text{ſin.}\, e : \text{ſin.}\, \frac{b+d}{2} \times \text{ſin.}\, \frac{b-d}{2} :: 2r : v.$$

And verſ. z — verſ. d : verſ. b — verſ. $d :: 2r : v$.
And the diff. of the verſed ſines is = diff. coſines.

Cor. 2. *Rectangle of the ſines of the ſides :*
Radius ſquare : :
Verſ. ſum of the ſides — verſ. baſe :
Verſ. ſup. of the included angle B.

For (by Cor. I)
Verſ. z — verſ. d : verſ. b — verſ. $d :: 2r : v$.
And by diviſion,
Verſ. z — verſ. d : verſ. z — verſ. $b :: 2r : 2r - v$.
Or verſ. z — verſ. $d : 2r ::$ verſ. z — verſ. $b : 2r - v$.
And $\frac{\text{verſ.}\, z - \text{verſ.}\, d}{2} \times r : rr ::$ verſ. z — verſ. $b : 2r - v$.

that is (by Cor. 2 Prop. 3, I) ſin. $a \times$ ſin. $e : rr ::$ verſ. z — verſ. $b : 2r - v$, the verſ. ſup. B, and the diff. verſed ſines = diff. coſines. Whence,

Cor. 3. *Verſ. ſum of the ſides — verſ. diff. ſides :*
Verſ. ſum ſides — verſ. baſe : :
Twice radius :
Verſ. ſupplement of the included angle B.

FIG. 46.

Cor. 4. *As rectangle of the ſines of the ſides :*
Radius ſquare : :
Or as ſum of the ſines × diff. ſines, of ½ ſum and ½ diff. [ſides :
Radius ſquare : :
So $\text{ſ}.\ \overline{\tfrac{1}{2}\,\textit{baſe} + \tfrac{1}{2}\,\textit{diff. ſides}} \times \text{ſ}.\ \overline{\tfrac{1}{2}\,\textit{baſe} - \tfrac{1}{2}\,\textit{diff. ſides}}$:
Sine ſquare of half the included angle : :
So $\text{ſ}.\ \tfrac{1}{2}\,\textit{ſum of the 3 ſides} \times \text{ſ}.\ \overline{\tfrac{1}{2}\,\textit{ſum 3 ſides} - \textit{baſe}}$:
Coſine ſquare of half the included angle B.

Let V = ſup. B, $s = \text{ſin.}\ \tfrac{1}{2}\text{B}$, $c = \text{coſ.}\ \tfrac{1}{2}\text{B}$. The reſt as before, then (by Prop. 2, 1) $v = \frac{2ss}{r}$, and $\text{V} = \frac{2cc}{r}$, and by Prop. 4, 1) $\frac{\text{verſ.}\ b - \text{verſ.}\ d}{2} \times r = \text{ſin.}\frac{b+d}{2} \times \text{ſin.}\frac{b-d}{2}$, and $\frac{\text{verſ.}z - \text{verſ.}b}{2} \times r = \text{ſin.}\frac{z+b}{2} \times \text{ſin.}\frac{z-b}{2}$, and (by Cor. 2 Prop. 4, 1) $\text{ſ.}\ a \times \text{ſ.}\ e = \overline{\text{ſ.}\ \frac{a+e}{2} + \text{ſ.}\ \frac{a-e}{2}} \times \overline{\text{ſ.}\ \frac{a+e}{2} - \text{ſ.}\ \frac{a-e}{2}} = \overline{\text{ſ.}\ \tfrac{1}{2}z + \text{ſ.}\ \tfrac{1}{2}d} \times \overline{\text{ſ.}\ \tfrac{1}{2}z - \text{ſ.}\ \tfrac{1}{2}d}$. Now by this Prop. $\text{ſin.}\ a \times \text{ſin.}\ e : rr :: \text{verſ.}\ b - \text{verſ.}\ d : v$ or $\frac{2ss}{r} :: \frac{\text{verſ.}\ b - \text{verſ.}\ d}{2} \times r$ or $\text{ſin.}\ \frac{b+d}{2} \times \text{ſin.}\ \frac{b-d}{2} : ss$.

Again, $a+e+b = z+b$, and $\frac{a+e+b}{2} - b = \frac{z-b}{2}$, and by Cor. 2, $\text{ſin.}\ a \times \text{ſin.}\ e : rr :: \text{verſ.}\ z - \text{verſ.}\ b : \text{V}$ or $\frac{2cc}{2} :: \frac{\text{verſ.}\ z - \text{verſ.}\ b}{2} \times r$ or $\text{ſ.}\ \frac{z+b}{2} \times \text{ſ.}\ \frac{z-b}{2} : cc$. Hence alſo, ſince $d = a - e$,

Cor. 5. *As rectangle of the ſines of the ſides :*
Radius ſquare : :
So ſ. ½ ſum of 3 ſides — one ſide, ſ. ½ ſum — other ſide :
Sine ſquare of half the included angle.
Or $\text{ſ.}\frac{\text{baſe+dif. ſides}}{2} \times \text{s.}\ \overline{\frac{\text{baſe+dif. ſides}}{2} - \text{dif. ſides}}$:
Sine ſquare of half the included angle.

FIG. 46. Cor. 6. *From half the ſum of the three ſides, ſubtract the baſe, and two ſides ſeparately; then*
As ſ. half ſum of the three ſides × ſ. firſt difference :
Rectangle of the ſines of the 2^d and 3^d differences ::
So radius ſquare :
Tan. ſquare of half the included angle.

For by Cor. 4. ſin. $\frac{z+b}{2} \times$ ſin. $\frac{z-b}{2}$: ſin. $\frac{b+d}{2} \times$ ſin. $\frac{b-d}{2}$ :: ($cc : ss$ ::) rr : tan. ſquare of half the angle, by Prop. 1, 1.

SCHOLIUM.

Any of theſe compound proportions may eaſily be reſolved into two ſimple ones. For example, in Cor. 4, ſay, ſin. $a : r$ :: ſin. $\frac{b+d}{2}$: to the ſine of a fourth P. And ſin. $e : r$ :: ſin. $\frac{b-d}{2}$: Q; then P × Q = ss required. And the ſame may be done for the next prop. and its corollaries.

PROP. XLIII.

42. *In any ſpherical triangle,*
As rectangle of the ſines of two angles, A, B *:*
Radius ſquare ::
Verſ. ſum of the including angles — verſ. ſup. 3^d angle :
Verſ. included ſide ::
And ſo verſ. third angle — verſ. ſup. ſum of the including angles :
Verſ. included ſide ::
And ſo coſ. diff. of the including angles + coſ. 3^d angle :
Verſ. ſup. included ſide ::
And ſo verſ. ſup. 3^d angle — verſ. diff. including angles :
Verſ. ſup. included ſide ::
And ſo verſ. ſup. diff. including angles — verſ. 3^d angle :
Verſ. ſup. included ſide, AB.

For let B be the greater angle, A the leſſer, AB the included ſide, and put FIG. 42.

$a =$ ſin. A, $m =$ coſ. A, $c =$ coſ. AB, $v =$ verſ. AB,
$b =$ ſin. B, $n =$ coſ. B, $d =$ coſ. F, $\mathrm{V} =$ verſ. ſup. AB.

Then (by Cor. 1 Pr. XXXIX) $\frac{mnr + drr}{ab} = c = r - v$, therefore $v = \frac{abr - mnr - drr}{ab}$; but (by Cor. 1 Pr. 5, 1) $\frac{mn - ab}{r} =$ coſ. $\overline{\mathrm{A} + \mathrm{B}} =$ (by Sch. 1, 1) $r -$ verſ. $\overline{\mathrm{A} + \mathrm{B}} =$ verſ. ſup. $\overline{\mathrm{A} + \mathrm{B}} - r$, and $d = r -$ verſ. F $=$ verſ. ſup. F $- r$, whence

$$v = \frac{\text{verſ. } \overline{\mathrm{A} + \mathrm{B}} - \text{verſ. ſup. F}}{ab} rr$$

$$= \frac{\text{verſ. F} - \text{verſ. ſup. } \overline{\mathrm{A} + \mathrm{B}}}{ab} rr.$$

Again, $\mathrm{V} = r + c = \frac{abr + mnr + drr}{ab}$. But (by Cor. 1, Pr. 6, 1) $\frac{ab + mn}{r} =$ coſ. $\overline{\mathrm{B} - \mathrm{A}} = r -$ verſ. $\overline{\mathrm{B} - \mathrm{A}} =$ verſ. ſup. $\overline{\mathrm{B} - \mathrm{A}} - r$. Therefore

$$\mathrm{V} = \frac{\text{coſ. } \overline{\mathrm{B} - \mathrm{A}} + \text{coſ. F}}{ab} \times rr$$

$$= \frac{\text{verſ. ſup. F} - \text{verſ. } \overline{\mathrm{B} - \mathrm{A}}}{ab} rr$$

$$= \frac{\text{verſ. ſup. } \overline{\mathrm{B} - \mathrm{A}} - \text{verſ. F}}{ab} rr.$$

Cor. 1. *As verſ. ſum of two angles — verſ. diff. :*
To twice radius ::
Or as coſ. difference of two angles — coſ. ſum :
To twice radius ::
So verſ. ſum of the including angles — verſ. ſup. third
Verſ. included ſide :: [*angle :*

So

FIG. 42.

So verſ. third angle — verſ. ſup. ſum of the including angles :
Verſ. included ſide : :
And ſo coſ. diff. including angles + coſ. third angle :
Verſ. ſup. included ſide : :
And ſo verſ. ſup. third angle — verſ. diff. including angles :
Verſ. ſup. included ſide : :
And ſo verſ. ſup. diff. incl. angles — verſ. third angle :
Verſ. ſup. included ſide. AB.

For (by Cor. 2 Pr. 3. 1) ſin. A × ſin. B =

$$\frac{\text{coſ.}\,\overline{B-A}-\text{coſ.}\,\overline{B+A}}{2}\times r=$$

$$\frac{\text{verſ.}\,\overline{B+A}-\text{verſ.}\,\overline{B-A}}{2}\times r.$$ And all the reſt follows from this prop.

Cor. 2. *As rectangle of the ſines of two angles,* A, B :
Radius ſquare : :
So ſine $\frac{1}{2}$ ſum of the including ∠s + $\frac{1}{2}$ ſup. 3^d ∠
× ſine $\frac{1}{2}$ ſum including ∠s — $\frac{1}{2}$ ſup. 3^d ∠ :
Sine ſquare of half the included ſide : :
And ſo ſine half ſum of : 3^d ∠, and ſup. ſum incl: ∠s
× ſine $\frac{1}{2}$ their diff. :
Sine ſquare of half the included ſide : :
So ſine $\frac{1}{2}$ ſup. 3^d ∠ + $\frac{1}{2}$ diff. incl. ∠s
× ſine $\frac{1}{2}$ ſup. 3^d ∠ — $\frac{1}{2}$ diff. incl. ∠s :
Coſ. ſquare of half the included ſide : :
So ſine $\frac{1}{2}$ ſup. diff. incl. ∠s + $\frac{1}{2}$ the 3^d ∠
× ſine $\frac{1}{2}$ ſup. diff. incl. ∠s — $\frac{1}{2}$ the 3^d ∠ :
Coſ. ſquare of half the included ſide, AB.

For ſince the diff. coſ. = diff. verſed ſines, therefore (by Cor. 1 Pr. 3, 1) $\frac{\text{verſ.}\,\overline{B+A}-\text{verſ. ſup.}\,F}{2}\times r$

$$=\text{ſine}\,\frac{B+A+\text{ſup.}\,F}{2}\times\text{ſine}\,\frac{B+A-\text{ſup.}\,F}{2}.$$

And

FIG. 42.

And $\frac{\text{verſ. F} - \text{verſ. ſup.}\,\overline{B+A}}{2} \times r =$

$\text{ſin.}\,\frac{F + \text{ſup.}\,\overline{B+A}}{2} \times \text{ſin.}\,\frac{F - \text{ſup.}\,\overline{B+A}}{2}$.

And $\frac{\text{verſ. ſup. F} - \text{verſ.}\,\overline{B-A}}{2} \times r =$

$\text{ſin.}\,\frac{\text{ſup. F} + B - A}{2} \times \text{ſin.}\,\frac{\text{ſup. F} + A - B}{2}$.

And $\frac{\text{verſ. ſup.}\,\overline{B-A} - \text{verſ. F}}{2} \times r =$

$\text{ſin.}\,\frac{\text{ſup.}\,\overline{B-A} + F}{2} \times \text{ſin.}\,\frac{\text{ſup.}\,\overline{B-A} - F}{2}$.

And (by Pr. 2, 1) verſ. AB × $\frac{1}{2}r$ = ſine ſquare of $\frac{1}{2}$AB, and verſ. ſup. AB × $\frac{1}{2}r$ = coſ. ſquare of $\frac{1}{2}$AB, then all the reſt follows from the preſent prop.

Cor. 3. *Rectangle of the ſines of two angles,* A, B :
Radius ſquare : :
Or *ſum of the ſines* × *diff. ſines, of* $\frac{1}{2}$ *ſum, and* $\frac{1}{2}$ *diff.*
Radius ſquare : : [*angles* :
So *coſ.* $\frac{1}{2}$ *ſum* × *coſ.* $\frac{1}{2}$ *diff. of the* 3^{d} ∠ *and ſum incl. an-*
Sine ſquare of half the included ſide : : [*gles* :
So *coſ.* $\frac{1}{2}$ *ſum* × *coſ.* $\frac{1}{2}$ *diff. of* 3^{d} ∠, *and diff. incl. an-*
Coſ. ſquare of half the included ſide, AB. [*gles* :

For (by Cor. 2 Pr. 4, 1) ſin. A × ſin. B = $\text{ſin.}\,\frac{B+A}{2} + \text{ſin.}\,\frac{B-A}{2} \times \text{ſin.}\,\frac{B+A}{2} - \text{ſin.}\,\frac{B-A}{2}$; and

ſin.

FIG. 42.

$$\text{ſin.}\,\frac{B+A+180-F}{2}\times\text{ſin.}\,\frac{B+A-180+F}{2}=$$

$$\text{ſin.}\,90+\frac{B+A-F}{2}\times\text{ſin.}\,\frac{B+A+F}{2}-90=$$

$$\text{coſ.}\,\frac{F-A-B}{2}\times\text{coſ.}\,180-\frac{B+A+F}{2}=$$

$$\text{coſ.}\,\frac{F-A-B}{2}\times\text{coſ.}\,\frac{F+A+B}{2}.$$

Again,

$$\text{ſin.}\,\frac{180-F+B-A}{2}\times\text{ſin.}\,\frac{180-F+A-B}{2}$$

$$=\text{ſin.}\,90-\frac{F+A-B}{2}\times\text{ſin.}\,90-\frac{F+B-A}{2}$$

$=\text{coſ.}\,\frac{F-B+A}{2}\times\text{coſ.}\,\frac{F+B-A}{2}$. The reſt follows from Cor. 2.

Cor. 4. From half the ſum of the three angles, ſubtract each angle, at the baſe ſeparately, and then the third angle. Then

As rectangle of the coſines of the two firſt differences :
Rectangle of the coſ. third difference and coſ. ½ *ſum of three angles ::*
So radius ſquare :
Tan. ſquare of half the included ſide or baſe, AB.

This follows from Cor. 3, becauſe coſ. : ſine :: rad : tangent. See ſchol. Prop. XLII.

PROP. XLIV.

48. *If the vertical angle* F *of a triangle* AFB *be biſected, it will be,*

As twice the cot. of the biſecting line FI :
Sum of the cotangents of the ſides AF, BF ::
Radius :
Coſ. half the vertical angle F.

Produce FA, FI, FB to quadrants, and to the FIG.
pole F draw the great circle LTR, and produce BA 48.
to interſect it in R. And let tan. AS $= b$, tan. IT $= c$, tan. BL $= d$, ſin. RT $= y$, coſ. $\frac{1}{2}$F $= x$. Then the angles at S, T, L are right. And by Cor. 1 Prop. XXVII, $c : b :: y : \frac{by}{c} =$ ſin. RS. And $c : d :: y : \frac{dy}{c} =$ ſin. RL. And ſince ST $=$ TL; therefore (by Prop. 3, 1) $\frac{by + dy}{c} r = 2x \times y$, or $\frac{b + d}{c} \times r = 2x$, that is, $2cx = \overline{b + d} \times r$.

Cor. *As twice radius* :
Tan. biſecting arch ::
Sum of the cotan. ſides :
Coſ. half the vertical angle F.

PROP. XLV.

If an arch be drawn from the vertex to the middle of the baſe of a triangle; then the ſines of the vertical angles are reciprocally as the ſines of the ſides, and directly as the ſines of the angles at the baſe. 48.

For (by Cor. 1 Prop. XXIX) ſin. AFI × ſin. AF = ſin. BFI × ſin. BF, becauſe AI = BI. Therefore ſin. AFI : ſin. BFI :: ſin. BF : ſin. AF :: (by Pr. XXIX) ſin. A : ſin. B.

PROP. XLVI.

As one right angle :
To the angle intercepted between two great circles :: 49.
So the area of a great circle of the ſphere :
Lunular area contained between theſe great circles.

Let

FIG. 49. Let ABC, ADC be the two great circles; BDF a great circle, whoſe poles are A and C. Divide the circle BDF into an infinite number of equal parts, through all which draw great circles paſſing through A and C, which will divide the ſurface of the ſphere into the ſame number of equal parts, ſimilar and equal to one another; ſince their baſes (in the circle BF) and ſides are all equal. Therefore as the number of the parts in the whole circumference, to the number of parts in BD, that is, as the whole circumference, to the arch BD : : ſo the ſurface of the ſphere, to the area ABCDA; or, as 4 right angles : angle BAD : : ſurface of the ſphere : area ABCDA. But (by geometry) the ſurface of the ſphere = 4 great circles. Therefore as 4 right angles : angle BAD : : 4 great circles : area ABCDA; or as one right angle : one great circle : : angle BAD : area ABCDA.

Cor. *As* 4 *right angles* :
Intercepted angle BAD : :
So ſurface of the ſphere :
Surface intercepted between theſe two great circles : :
And ſo ſolidity of the ſphere :
Solidity contained between theſe great circles.

For the ſolidities are as the ſurfaces, becauſe they may be reſolved into pyramids of equal height.

PROP. XLVII.

50. *In any ſpherical triangle*, G;
As two right angles :
Exceſs of the three angles above two right angles : :
Area of a great circle of the ſphere :
Area of the triangle.

Produce all the ſides to ſemicircles, then the oppoſite angles are equal (by Cor. 1 Prop. IV); and (by

FIG. 50.

(by Prop. IX) triangle $H =$ triangle G, alſo half the ſurface of the ſphere $= G + R + S + T$; let a, b, c be the three angles, then (by Cor. Prop. XLVI),

$180° : a :: \frac{1}{2}$ ſurface ſphere $: G + R$,
$180 : b :: \frac{1}{2}$ ſurface ſphere $: G + S$;
$180 : c :: \frac{1}{2}$ ſurface ſphere $: H + T$ or $G + T$,

therefore $180 : a + b + c :: \frac{1}{2}$ ſurface ſphere $: 3G + R + S + T$, and by diviſion, $180 : a + b + c - 180 :: \frac{1}{2}$ ſurface ſphere $: 2G :: \frac{1}{4}$ ſurface ſphere, or the area of a great circle $: G$.

Cor. 1. *As* $1 : 57.2957795 ::$ *the* 3 *angles* — 2 *right angles* : *area, in ſquare degrees.*

For area of a great circle $= \frac{1}{2}$circumference × radius $= 180 \times$ radius, and radius $= 57.2957795$ &c.

Cor. 2. *As* $180 : 3$ *angles* $- 180 :: 3.1416 \times \overline{rad.}^2$: *area of the triangle.*

For $3.1416rr =$ area of the circle, whoſe radius is r.

Cor. 3. *In any ſpherical polygon, put* $n =$ *number of ſides,* $A = 180° \times \overline{n - 2}$. Then I ſay,

As 2 *right angles or* 180° :
Sum of all the angles of the polygon $- A$ ::
So the area of a great circle of the ſphere :
Area of the polygon.

Suppoſe the polygon to be divided into as many triangles as it has ſides, by arches drawn from a point within it. Then ſince (by this Prop.) it is in any one triangle, as $180 :$ a great circle :: ſum of its angles $- 180$: its area; therefore by compoſition it will be, $180 :$ a great circle :: ſum of all the angles internal and external $- n \times 180$: ſum of all the areas of the triangles; but the ſum of the internal angles =

FIG. $=360$ or 2×180, therefore 180 : great circle :: ſum of all the angles of the polygon — A (or $\overline{n-2} \times 180$: area of the triangle.

PROP. XLVIII.

71. *If a ſpherical triangle* ABG *be inſcribed in a leſſer circle, ſo as that the baſe* AG *paſs through the pole* D *of that circle; then the angle at the vertex* ABG *will be equal to the ſum of the angles at the baſe* A, G; *and the contrary.*

For draw BD to the pole D. Then ſince AD, DB, DG are all equal, the angle A $=$ ABD, and the ∠ G $=$ DBG; and therefore the ſum A $+$ G $=$ ABG.

And if the ∠ B $=$ A $+$ G, then AD $=$ DB $=$ DG. For if DB be greater or leſſer than AD or DG, then A will be greater or leſſer than ABD, and G greater or leſſer than DBG; and therefore A $+$ G accordingly greater or leſſer than ABG.

PROP. XLIX.

72. *If the ſum of the two ſides* (AB, BG) *of the ſpherical triangle* ABG, *be a ſemicircle, and the included angle* B *be biſected; the oppoſite ſide* AG *will alſo be biſected, and the biſecting arch* BD *will be a quadrant.*

For produce BA, BD, BG to E, then BE is a ſemicircle $=$ AB $+$ BG; and EG $=$ AB. Alſo angle ABD $=$ DBG $=$ DEG (by Prop. V), and ∠ BAD $=$ EGD (by Prop. XVI); and the alternate angles BDA and and EDG are equal; therefore in the two triangles BAD, and EGD, all the angles are reſpectively equal; whence (by Prop. XI) AD $=$ DG, and BD $=$ DE, half a ſemicircle.

PROP. L.

If from the vertex of any ſpherical triangle ABG, *two arches*, BD, BE *be drawn to cut the baſe* AG, *making the angles* ABD, GBE *equal*; *it will be*, FIG. 73.

As ſquare of the ſine of AB :
To ſquare of the ſine of BG : :
So rectangle of the ſines of AD, AE :
To rectangle of the ſines of GE, GD.

For ſince the $\angle$ ABD $=$ GBE, therefore ABE $=$ DBG. And (by Cor. 1 Prop. XXIX)

ſin. AB $\times$ ſin. ABD : ſin. BG $\times$ ſin. DBG or ABE : : ſin. AD : ſin. DG.

And ſin. AB $\times$ ſin. ABE : ſin. BG $\times$ ſin. EBG or ABD : : ſin. AE : ſin. EG. Therefore

$\overline{\text{ſin. AB}}^2 : \overline{\text{ſin. BG}}^2 :: \text{ſin. AD} \times \text{ſin. AE} : \text{ſin. DG} \times \text{ſin. EG}$.

PROP. LI.

In any ſpherical triangle ABG, *if two angles* A, G *be biſected by the arches* AD, GD ; *and if an arch* BDE *be drawn from the third angle* B, *to the point of concourſe* D, *it will alſo biſect that angle.* 74.

For (by Cor. 2 Prop. XXIX) ſin. BD : ſin. DE : : ſin. BG : ſin. GE : : ſin. BA : ſin. AE ; and alternately ſin. BG : ſin. BA : : ſin. GE : ſin. AE : : ſin. BG $\times$ ſin. GBE : ſin. BA $\times$ ſin. ABE (by Cor. 1 ib), therefore $\angle$ GBE $=$ ABE.

Cor. *Hence, three arches biſecting the three angles of a ſpherical triangle, interſect each other in one point.*

PROP. LII.

FIG. 74. *In any ſpherical triangle* ABG, *if two arches* AC, GF *be drawn from two angles* A, G, *to biſect the oppoſite ſides* BG, AB; *then the arch* BDE, *drawn from the third angle* B *through the point of concourſe* D, *will alſo biſect the third ſide* AG.

For by Prop. XXIX and Cor. 1, we ſhall have

ſin. AG × ſin. AGD = ſin. AD × ſin. ADF = ſin. BD × ſin. BDF, becauſe AF = FB. Alſo

ſin. AG × ſin. GAD = ſin. GD × ſin. GDC = ſin. BD × ſin. BDC, ſince BC = GC. Therefore

ſin. GAD : ſin. AGD :: ſin. BDC or ſin. ADE : ſin. BDF or GDE.

But ſin. AE : ſin. EG :: ſin. AD × ſin. ADE : ſin. GD × ſin. GDE :: ſin. AD × ſin. GAD : ſin. GD × ſin. AGD. But in the triangle ADG, ſin. AD × ſin. GAD = ſin. GD × ſin. AGD; therefore AE = EG.

Cor. *Three arches drawn from the three angles of a ſpherical triangle to biſect the oppoſite ſides, will meet in one point.*

PROP. LIII.

75. *In any ſpherical triangle* ABG, *if three arches be erected perpendicularly upon the middle of the three ſides; they will interſect in one point.*

Let CD, FD be two perps. draw GD, BD, AD; then in the right-angled triangles CDG, CDB; BC = CG, and CD common; therefore GD = DB. Alſo in the right-angled triangles BDF, ADF, we have BF = AF, and FD common; therefore AD = DB = GD. Therefore the triangle ADG is iſoſceles, and therefore the arch DE, which biſects the baſe AG, is perpendicular to it.

PROP. LIV.

In any ſpherical triangle ABG, *if two arches be drawn from two angles* A, G, *perpendicular to the oppoſite ſides* GB, AB; *and if the arch* BDE *be drawn from the third angle* B *through the point of interſection* D, *it will be perpendicular to the third ſide* AG. FIG. 76.

The ratio of ſin. AD to ſin. DE is compounded of ſin. AD to ſin. DC, ſin. DC : to ſin. DB, and ſin. DB to ſin. DE. And by Prop. XXIX Cor. 1, in the triangles AGC, CBD, BGE, it is,

ſ. AD : ſ. DC :: ſ. AG × ſ. AGD : ſ. CG × ſ. CGD, and
ſ. DC : ſ. DB :: ſ. CBD : radius 1.
ſ. DB : ſ. DE :: ſ. BG × ſ. BGD : ſ. EG × ſ. EGD,
and throwing out the equal ratios ſ. BGD, ſ. CGD, and ſ. AGD, ſ. EGD, then
ſ. AD : ſ. DE :: ſ. AG × ſ. BG × ſ. CBD : ſ. CG × ſ. EG.

But in the triangle BGE, ſ. BG × ſ. CBD or ſ. GB × ſ. GBE = ſ. GE × ſ. GED.

Therefore ſ. AD : ſ. DE :: ſ. AG × ſ. GE × ſ. GED : ſ. CG × ſ. EG :: ſ. AG × ſ. GED : ſ. CG.

But ſ. AG : ſ. CG :: 1 rad. : ſ. CAG, in the triangle ACG; therefore ſ. AD : ſ. DE :: ſ. GED : ſ. CAG or EAD.

But in the triangle ADE; ſ. AD : ſ. DE :: ſ. AED : ſ. EAD.

Therefore ſ. GED : ſ. EAD :: ſ. AED : ſ. EAD, or ſ. GED : ſ. AED :: ſ. EAD : ſ. EAD, therefore GED = AED, or BDE perpend. to AG.

Cor. *Three perpendiculars let fall from the three angles to the oppoſite ſides, all meet in one point.*

PROP. LV.

FIG. 77. *In the right-angled ſpherical triangle* ABG; *if from the right angle* ABG, *two arches* BD, BE, *be drawn, making equal angles with the ſide* BA; *then will* ſin. GD : ſin. GE :: ſin. AD : ſin. AE.

For ſince the ∠ ABG is right, and EBA = ABD, the ∠ EBG + DBG = 2 right angles, or EBG = ſup. of DBG. Therefore in the two triangles GBD, GBE, it is ſin. GD : ſin. DB :: ſin. GBD or GBE : ſin. G :: ſin. GE : ſin. EB, by Prop. XXIX, and alternately, ſin. GD : ſin. GE :: ſin. DB : ſin. EB :: ſin. DA : ſin. EA, by Cor. 1 Prop. XXIX.

PROP. LVI.

78. *In any triangle* LMN, *if any ſide* NM *be produced, and the arch* EOA *be drawn to interſect the three ſides of the triangle, in the points* A, O, E; *it will be,* ſ. LA : ſ. AN :: ſ. LO × ſ. EM : ſ. OM × ſ. EN.

For in the triangle AEN, ſin. A : ſin. EN :: ſin. E : ſin. AN $= \frac{\text{ſin. E} \times \text{ſin. EN}}{\text{ſin. A}}$, and in the triangle LAO, ſin. A : ſin. LO :: ſin. O : ſin. LA $= \frac{\text{ſin. O} \times \text{ſin. LO}}{\text{ſin. A}}$. Therefore ſin. LA : ſin. AN :: ſin. LO × ſin. O : ſin. EN × ſin. E :: ſin. LO × ſin. EM : ſin. EN × ſin. OM, in the triangle OEM.

PROP. LVII.

If an arch EOA *be drawn to interſect the three ſides of a triangle* LMN *in the points* A, O, E; *any ſide thereof being produced; then will* FIG. 78

ſ. NL : ſ. NA : : ſ. LM × ſ. EO : ſ. AE × ſ. OM.

For in the triangle NLM; ſin. N : ſin. LM : : ſin. M : ſin. NL $= \frac{\text{ſin. LM} \times \text{ſin. M}}{\text{ſin. N}}$. And in the triangle AEN; ſin. N : ſin. AE : : ſin. E : ſin. AN $= \frac{\text{ſin. AE} \times \text{ſin. E}}{\text{ſin. N}}$. Whence ſin. NL : ſin. AN : : ſin. LM × ſin. M : ſin. AE × ſin. E : : ſin. LM × ſin. EO : ſin. AE × ſin. OM; becauſe ſin. M : ſin. E : : ſin. EO : ſin. OM, in the triangle OEM.

SECT. IV.

The ſolution of all the caſes of ſpherical triangles.

FIG. By help of the propoſitions delivered in the laſt ſection, all the caſes of ſpherical triangles may be reſolved. I ſhall give the canons for the ſolution of each caſe in particular, both in right-angled triangles, and in oblique triangles; after I have ſhewn, in general, how they all may be reſolved by the foregoing propoſitions.

Every triangle has ſix parts, and any three being given, the reſt may be found.

The ſolution of right-angled triangles.

All the caſes of right-angled and quadrantal triangles may be reſolved by the XXVIth and XXVIIth propoſitions; or by Prop. XXVIII alone.

I.

41. The ſolution of any caſe may be performed by the XXVIth or XXVIIth Prop. one of the two; either immediately in the triangle itſelf, or elſe in another right-angled triangle, formed by producing all its ſides to quadrants; wherein the parts of the triangle are either the ſame, or the complements of thoſe in the firſt. Thus let ABF be the propoſed triangle; produce the ſides either through A or F, ſo that AC, AD, BE, and conſequently CE, may be quadrants. Then you have two right-angled triangles, ABF, EDF. In the figure ABCDEF, mark what is given, and its complement, with a daſh (/), and what is ſought, and its complement, with a cypher (o). Then examine the two right-angled triangles ABF, EDF; and obſerve in which of them, *an angle and*

41. *the hypothenuſe, or an angle and its oppoſite ſide,* do not enter the queſtion, (for there are always two parts out of the queſtion); and that triangle will afford the ſolution, by applying Prop. XXVII in the former caſe, and Prop. XXVI in the latter. If any ſide be greater than a quadrant, then a quadrant muſt be cut off, and the ſame rule applied.

If the ſolution is neither found in ABF nor EDF; produce the ſides from, thro' A and B to quadrants as before, to find a third triangle, and then the ſolution will be found in this new complemental triangle.

Quadrantal triangles may be reſolved by the ſame propoſitions, if you reduce them to right-angled triangles, by producing their oblique ſides to quadrants, or cutting off a quadrant if they be bigger. As in the quadrantal triangle AFE, produce AF, EF to quadrants, as AD, EB, and draw AB, ED round the poles E, A; then you will have two right-angled triangles ABF, EDF, to be reſolved by the former rule. Take an example or two for right-angled triangles.

Ex. 1.

51. Let the hypothenuſe AF and an angle F be given, and the oppoſite ſide AB required. I mark AF, F as given, and AB as ſought; then ſince there is an angle A and its oppoſite ſide FB out of the queſtion, therefore I need produce the ſides no further, for I have the ſolution in the triangle ABF, by Prop. XXVI, as rad. : ſin. AF : : ſin. F : ſin. AB required.

Ex. 2.

41. Let the hyp. AF, angle A be given; to find the other angle F. I produce the ſides through F, and mark AF, FD, A, DC, ED as given; and F as ſought. Then I ſee, that in the triangle EFD, the angle E and hypothenuſe EF are not marked, and therefore by Prop. XXVII, ſin. FD : rad. : : tan. ED : tan. angle F; that is, coſ. AF : rad. : : cotan. A : tan. F required.

FIG.

II.

Right-angled or quadrantal triangles may alſo be reſolved by Prop. XXVIII, obſerving if middle part be ſought, to begin with radius; and if one extreme be ſought, to begin with the other extreme.

Ex. 1.

51. Given the hyp. AF, and angle F, to find the ſide AB; what is given and ſought being marked as in the figure. Then AB is middle part, and AF, F, extremes disjunct, therefore rad. : coſ. comp. AF : : coſ. comp. F : ſin. AB; that is, rad. : ſin. AF : : ſin. F : ſin. AB ſought.

Ex. 2.

41. Given the hyp. AF, and an angle A, to find the angle F. Here AF is middle part, and A, F extremes conjunct. Therefore tan. comp. A : rad. : : ſin. compl. AF : tan. comp. F, or cotan. A : rad. : : coſ. AF : cotan. F.

Ex. 3.

41. In the quadrantal triangle AFE, let ſide AF, angle A, be given; to find the angle F. Here AF is middle part, and A and F are extremes conjunct; therefore,

Tan. EAF : rad. : : coſ. AF : cotan. F.

Laſtly, the affection of the angle or ſide ſought will be known by Prop. XX, and Prop. XXI and its corollaries; except it be ambiguous, and then it may be either acute or obtuſe.

The ſolution of oblique ſpherical triangles.

Oblique triangles may be reſolved ſeveral ways, either by letting fall a perpendicular, or without it. In any triangle, mark what is given with a daſh (/), and what is required with a cypher (o), to diſtinguiſh them.

I.

Oblique triangles may be reſolved by the rules of right-angled triangles, together with the corollaries of Prop. xxviii, with Prop. xxix, xxxii, xxxiii; by letting fall a perpendicular, which either divides the triangle into two right-angled triangles; or makes two right-angled triangles, by adding a right-angled triangle to it.

In letting fall the perpendicular, let it fall from the end of a given ſide, and oppoſite to a given angle; and if the three things given be adjoining to one another, let it alſo fall from the end of the ſide required, or oppoſite to the angle required; and then you will have enough given in one of the right-angled triangles, to find any of its unknown parts: obſerving to find ſuch a part (which muſt be either a baſe or vertical angle), as either of itſelf, or when added to or ſubtracted from ſome part given, the proportion to find the thing ſought may come under ſome of the corollaries of Prop. xxviii.

II.

All the caſes of oblique triangles may alſo be reſolved by Prop. xxvii, xxix, xxxii, and xxxiii, alone, by letting fall a perpendicular according to the former directions, when there is occaſion. For the two laſt caſes will be reſolved by Prop. xxxiii and xxxii, and all the reſt by Prop. xxvii and xxix; and they are eaſily remembered, for three of them are the ſame with thoſe in plain trigonometry, for reſolving the like caſes. Only there will be more operations in the right-angled triangles, than if the corollaries of Prop. xxviii were made uſe of.

III.

All the caſes of oblique triangles (except the 1ſt, 4th, and two laſt) may alſo be reſolved by Prop. xxviii alone, by letting fall a perpendicular dividing the whole into two right-angled triangles. In the firſt

FIG. firſt triangle you have two things given to find a third, which muſt be either the baſe or vertical angle. In the ſecond triangle you have one thing given, or at leaſt you can find one thing, after the firſt operation in the firſt triangle is over; and in the ſecond triangle there is alſo the thing ſought. Then in theſe two triangles you have either both the baſes, or elſe both the vertical angles. Therefore, from theſe, to find the thing required, compare theſe three things in the ſecond triangle, viz. the perpendicular, the thing given, and the thing ſought, and find which is middle part, and whether the other two be extremes conjunct or disjunct, in order to know whether you muſt uſe ſines or tangents. Do the ſame thing in the firſt triangle, with the three correſponding parts; and (ſetting aſide the perpendicular, which is an extreme in both triangles); then the middle part and its extreme in the firſt triangle, will be proportional to the middle part and its extreme in the ſecond triangle.

53. *For example.* Given FB, F and B; to find AF. In the firſt triangle BFI, as cotan. B : rad. : : coſ. FB : cotan. BFI; then you have AFI. Then in the triangle AFI, the angle AFI is middle part, and AF, FI extremes conjunct. Likewiſe in the triangle BFI; BFI is middle part, and BF, FI extremes conjunct. Therefore coſ. BFI (middle part) : coſ. AFI (middle part) : : cotan. BF (extreme conjunct) : cotan. AF (extreme conjunct.)

The reaſon of this operation will appear thus: let $p =$ perpendicular, $m =$ middle part, $a =$ the other extreme, in the firſt triangle; $M =$ middle part, $A =$ extreme in the ſecond triangle. Then, by Prop. XXVIII, $pa = mr$, and $pA = Mr$, therefore $\frac{p}{r} = \frac{m}{a} = \frac{M}{A}$, or $m : a :: M : A$.

IV.

All oblique triangles may alſo be reſolved without letting fall a perpendicular, and theſe five propoſitions,

poſitions, the XXIX, XXXVI, XXXVII, XL, XLI, and their corollaries, will reſolve them all. Thus, Caſe 1 and 4 are reſolved by Prop. XXIX, Caſe 2 and 6 by Prop. XXIX, and Cor. 1. Prop. XXXVII. Caſe 3 and 5 by Prop. XXIX, and Cor. 1 Prop. XXXVII. Caſe 7 by Prop. XXXVI and Cor. 1 Caſe 8 by Prop. XXXVI, and Cor. 1, and Prop. XXIX. Caſe 9 by Prop. XXXVII, and Cor. 1. Caſe 10 by Prop. XXXVII and Cor. 1, and Prop. XXIX. Caſe 11 by Prop. XLII, or its corollaries. Caſe 12 by Prop. XLIII, or its corollaries. FIG. 53.

Laſtly, the affection of the angle or ſide ſought may be gathered from Prop. XVII, and its corollaries. And the falling of the perpendicular, from Prop. XIX, and Cors. Prop. VIII.

Here follow the 16 caſes of right-angled triangles, and the 12 of oblique. In right-angled triangles, I ſhall give only the ſolution by the tables, omitting the algebraic ſolution; for that is very eaſily had by putting letters inſtead of the quantities, and tranſforming them by Sch. Prop. 1, 1, if there is occaſion.

In oblique triangles you have both; where note, that natural ſines, tangents, &c. muſt be uſed in the algebraic way. I refer to the caſes in right-angled triangles, for the determination of the ſpecies of the angle or ſide ſought, in the method of letting fall a perpendicular.

The rule for working any log. proportion in ſpherical trigonometry, is the ſame as in plain trigonometry, viz. *Add the logarithms of the ſecond and third terms together, and from the ſum ſubtract the log. of the firſt term; and there remains the log. of the fourth term ſought.*

Note, All proportions of ſpherical triangles agree with plain triangles, taking the ſides inſtead of ſines and tangents of the ſides. But then neither coſines, cotangents, coſecants, or verſed ſines of the ſides muſt be concerned. And radius can only come in when two terms are ſides.

Right-

Right-angled ſpherical triangles.

CASE I.

FIG. 52. Given the *hypothenuſe* AF, and an *angle* A; to find the *ſide adjacent* AB.

Rad. : *coſ.* angle A : : *tan. hyp.* AF : *tan. adjacent ſide* AB.

If the angle and hyp. (A, AF) are of the ſame affection, AB is leſs than a quadrant; if of different affection, more.

CASE II.

52. Given the *hyp.* AF, and *angle* A; to find the oppoſite *ſide* BF.

Rad. : ſin. *hyp.* AF : : ſin. *an angle* A : ſin. *oppoſite ſide* BF.

If the angle A be acute, BF is leſs than 90; if obtuſe, more.

CASE III.

2. Given the *hyp.* AF, and an *angle* A; to find the other *angle* F.

Rad. : *coſ. hyp.* AF : : *tan. an angle* A : *cotan. other angle* F.

When the hyp. and angle (AF and A) are of the ſame affection, F is acute; if of different affection, obtuſe.

CASE IV.

52. Given the *hyp.* AF, and a *leg* AB; to find the adjacent *angle* A.

Rad. : *cot. hyp.* AF : : *tan. a ſide* AB : *coſ. adjacent angle* A.

If the hyp. and leg (AB, AF) be of the ſame affection, A is acute; if of different affection, obtuſe.

CASE V.

Given the *hyp.* AF, and one *ſide* AB; to find the oppoſite *angle* F. FIG. 52.

Sin. *hyp.* AF : *rad.* :: ſin. *one ſide* AB : ſin. its *oppoſite angle* F.

If the ſide (AB) is leſs than 90, F is acute; if more than 90, obtuſe.

CASE VI.

Given the *hyp.* AF, and a *leg* AB; to find the other *leg* BF. 52.

Coſ. one leg AB : *coſ. hyp.* AF :: *rad.* : *coſ. other leg* BF.

If the hyp. and leg (AB, AF) be of one affection, BF is leſs than 90; if of different affection, more.

CASE VII.

Given a *ſide* AB, and its adjacent *angle* A; to find the oppoſite *ſide* FB. 52.

Rad. : ſin. *one ſide* AB :: *tan. adjacent angle* A : *tan. oppoſite ſide* FB.

If the angle (A) be acute, BF is leſs than 90; if obtuſe, greater.

CASE VIII.

Given a *ſide* AB, and its adjacent *angle* A; to find the oppoſite *angle* F. 52.

Rad. : ſin. *an angle* A :: *coſ. its adjacent ſide* AB : *coſ. its oppoſite angle* F.

If the ſide (AB) is leſs than 90, F is acute; if more, obtuſe.

CASE

CASE IX.

FIG. 52. Given a *ſide* AB, and *angle adjacent* A; to find the *hyp.* AF.

Coſ. an angle A : *rad.* : : *tan. adjacent ſide* AB : *tan. hyp.* AF.

If the angle and ſide (A and AB) be of the ſame affection, AF is leſs than 90; if different, more.

CASE X.

52. Given a *ſide* BF, and oppoſite *angle* A; to find the *ſide* adjacent AB.

Rad. : *cotan. an angle* A : : *tan. op. ſide* BF : ſin. *adjacent ſide* AB.

Here AB is ambiguous, that is, it may be either leſſer or greater than 90.

CASE XI.

52. Given a *ſide* BF, and oppoſite *angle* A; to find the *angle* adjacent F.

Coſ. one ſide BF : *coſ. op. angle* A : : *rad.* : ſin. *adjacent angle* F, ambiguous.

CASE XII.

52. Given a *ſide* BF, and *op. angle* A; to find the *hyp.* AF.

Sin. *an angle* A : ſin. *op. ſide* BF : : *rad.* : ſin. *hyp.* AF, ambiguous.

CASE XIII.

52. Given the *ſides* AB, FB, to find an *angle* A.

Sin. *adjacent ſide* AB : *rad.* : : *tan. other ſide* FB : *tan. op. angle* A.

If the oppoſite ſide (BF) is leſs than 90, A is acute; if more, obtuſe.

CASE

CASE XIV.

Given the *ſides* AB, FB; to find the *hyp.* AF. FIG. 52.

Rad. : *coſ. one leg* FB :: *coſ. other leg* AB : *coſ. hyp.* AF.

If the ſides (AB, FB) are of one affection, AF is leſs than a quadrant; if of different affection, more.

CASE XV.

Given the *angles* A, F; to find a *ſide* BF. 52.

Sin. *adjacent angle* F : *rad.* :: *coſ. other angle* A : *coſ. op. ſide* BF.

If the oppoſite angle (A) be acute, BF is leſs than a quadrant; if obtuſe, it is greater.

CASE XVI.

Given the *angles* A, F; to find the *hyp.* AF. 52.

Tan. one angle F : *cotan. other angle* A :: *rad.* : *coſ. hyp.* AF.

If the angles (A, F) be of the ſame affection, AF is leſs than a quadrant; if of different affection, it is greater.

Oblique ſpherical triangles.

CASE I.

FIG. 54. 55. 56. Given two *ſides* AF, BF, and an oppoſite *angle* A; to find the other oppoſite *angle* B.

I. *Logarithmically, by the table of artificial ſines,* &c.

Sin. *one ſide* FB : ſin. *op. angle* A :: ſin. other *ſide* AF : ſin. *its op. angle* B.

Then $\frac{A+B}{2}$ is of the ſame affection as $\frac{AF+BF}{2}$.

And if $\frac{A+B}{2}$ be of the ſame affection as $\frac{A+\text{ſup.}B}{2}$, then B is ambiguous.

II. *Algebraically, by the table of natural ſines,* &c.

Let $d =$ ſin. AF, $p =$ ſin. BF, $a =$ ſin. A; then ſin. B $= \frac{ad}{p}$.

CASE II.

54. 55. 56. Given two *ſides* AF, BF, and an oppoſite *angle* A; to find the included *angle* F.

I. *Logarithmically.*

Let fall a perpendicular from F, (beyond B, fig. 56.) Then by Caſe 3. right ∠ triangles, R : *coſ.* AF :: *tan.* A : *cotan.* AFI.

And *cotan.* AF : *coſ.* AFI :: *cotan.* BF : *coſ.* BFI.

Then according as BF and angle FAI are of the ſame or different affection, BFI is acute or obtuſe.

If

$$\left.\begin{array}{l}\text{If } AFI > BFI, \text{ then } AFI - BFI = F\\ \text{If } AFI + BFI < 180, \text{ then } AFI + BFI = F\end{array}\right\}$$ if both, then F is ambiguous. FIG. 54. 55. 56.

Or thus,

Find the angle B *by Caſe* 1; *then*

$$\text{ſin. } \frac{AF \backsim BF}{2} : \text{ſin. } \frac{AF + BF}{2} :: \text{tan. } \frac{B \backsim A}{2} : \text{cot. } \tfrac{1}{2}F.$$

II. *Algebraically.*

Let a, d, p *be the ſines, and* m, e, i *the coſines of* A, AF, BF. *And* $b = \frac{ad}{p} =$ ſin. B, $n =$ *coſ.* B, *and* $bb = bb - aa = mm - nn$, $kk = 1 - bbpp = 1 - aadd$, *rad.* $= 1$. 47.

Then ſin. $F = \frac{bb}{bim - aen} = \frac{bim + aen}{kk}$.

Alſo ſee Cor. 1, 2 *Prop.* XL.

CASE III.

Given two *ſides* AF, BF, and an oppoſite *angle* A; to find the third *ſide* AB. 54. 55. 56.

I. *Logarithmically.*

Let fall a perp. from F, then *by Caſe* 1 *right* ∠ *triangles,*

Rad. : *coſ.* A :: *tan.* AF : *tan.* AI.

Then *coſ.* AF : *coſ.* AI :: *coſ.* BF : *coſ.* BI.

Then if BF and angle FAI are of one affection, BI is leſs than a quadrant; if of different affection, more.

FIG. If $AI > BI$, $AI - BI = AB$, } if both, then
54. If $AI + BI < 180$, $AI + BI = AB$, }
55. AB is ambiguous.
56.

Or thus,

Find the angle B *by Caſe* 1; *then*

ſin. $\frac{B \backsim A}{2}$: ſin. $\frac{B + A}{2}$: : *tan.* $\frac{AF \backsim BF}{2}$: *tan.* $\frac{1}{2}AB$.

II. *Algebraically.*

Let a, d, p *be the ſines, and* m, e, i *the coſines of* A, AF, BF, $b = \frac{ad}{p}$ = ſin. B, n = *coſ.* B, *and* $ll = dd - pp = ii - ee$, $tt = 1 - bbpp = 1 - aadd$, *radius* $= 1$. *Then*

$$\text{ſin. } AB = \frac{ll}{mid - pen} = \frac{mid + pen}{tt}.$$

See alſo Cor. 1, 2 *Prop.* XLI.

CASE IV.

54. Given two *angles* A, B, and an oppoſite *ſide* BF;
55. to find the other oppoſite *ſide* AF.
56.

I. *Logarithmically.*

Let fall a perp. from F, then
ſin. A : ſin. BF : : ſin. B : ſin. AF.

$\frac{AF + BF}{2}$ is of the *ſame affection* as $\frac{A + B}{2}$. If $\frac{BF + AF}{2}$ be of the *ſame affection* as $\frac{BF + \text{ſup.}\,AF}{2}$, then AF is ambiguous.

II. *Al-*

II. *Algebraically.*

Let $a =$ ſin. A, $b =$ ſin. B, $p =$ ſin. BF, FIG.
then ſin. AF $= \frac{pb}{a}$. 54. 55. 56.

CASE V.

Given two angles A, B, and an oppoſite *ſide* BF; to find the included *ſide* AB. 54. 55. 56.

I. *Logarithmically.*

Let fall a perp. from F, then by Caſe I right ∠ triangles, *rad.* : *tan.* BF : : *coſ.* B : *tan.* BI.

Then *cotan.* B : *cotan.* A : : ſin. BI : ſin. AI.

If A, B be of the *ſame affection*, AI + BI = AB. If BI + AI, and BI + ſup. AI be each < 180, (AI and conſequently) AB is ambiguous.

If A and B are of *different affection*, AI — BI = AB. If both AI and ſup. AI be > BI, then AI and AB are ambiguous.

Or thus,

Find AF *by Caſe* IV; *then* 47.

$$\text{ſin.}\,\frac{B \sim A}{2} : \text{ſin.}\,\frac{B + A}{2} :: tan.\,\frac{AF \sim BF}{2} : tan.\,\tfrac{1}{2}AB.$$

II. *Algebraically.*

Let a, b, p *be the ſines,* m, n, i *the coſines of* A, B, BF; $d = \frac{pb}{a} =$ ſin. AF, $e =$ coſ. AF, $ll = dd - pp = ii - ee$, $tt = 1 - bbpp = 1 - aadd$, rad. $= 1$. Then

FIG. 47. ſin. AB $= \frac{ll}{mid - pen} = \frac{mid + pen}{tt}$.

See alſo Cor. 1, 2 *Prop.* XLI.

CASE VI.

54. 55. 56. Given two *angles* A and B, and an oppoſite *ſide* BF; to find the third *angle* F.

I. *Logarithmically.*

Let fall a perp. from F; then by Caſe 3 right ∠ triangles, *rad.* : *tan.* B : : *coſ.* BF : *cotan.* BFI.

Then *coſ.* B : ſin. BFI : *coſ.* A : ſin. AFI.

If A and B are of the *ſame affection*, BFI + AFI = F. If both BFI + AFI and BFI + ſup. AFI be < 180, then (AFI and) F is ambiguous.

If A and B are of *different affection*, AFI — BFI = F, if both AFI and ſup. AFI > BFI, then (AFI and) F is ambiguous.

Or thus,

47. Find AF by Caſe 4; then ſin. $\frac{\text{AF} \backsim \text{BF}}{2}$: ſin. $\frac{\text{AF} + \text{BF}}{2}$: : *tan.* $\frac{\text{B} \backsim \text{A}}{2}$: *cotan.* $\frac{1}{2}$F.

II. *Algebraically.*

47. Let a, b, p be the *ſines*; m, n, i the *coſines* of A, B, FB; $d = \frac{pb}{a} =$ ſin. AF, $e =$ *coſ.* AF; $bb = bb - aa = mm - nn$, $kk = 1 - bbpp = 1 - aad$, rad. $= 1$.

Then ſin. F $= \frac{bb}{bim - aen} = \frac{bim + aen}{kk}$.

Alſo ſee Cor. 1, 2. Prop. XL.

CASE

CASE VII.

Given two *ſides* AF, AB, and the included *angle* A; to find an oppoſite *angle* B. FIG. 54. 55. 56.

I. *Logarithmically.*

Let fall a perp. from F; then by Caſe I right ∠ triangles, *rad.* : *coſ.* A :: *tan.* AF : *tan.* AI, and $AB \backsim AI = IB$.

Then ſin. AI : ſin. BI :: *cotan.* A : *cotan.* B.

According as AI is < or > than AB, B and A are of the ſame or different affection.

Or thus,

Coſ. $\frac{AB + AF}{2}$: *coſ.* $\frac{AB \backsim AF}{2}$ :: *cotan.* $\frac{1}{2}A$: *tan.* $\frac{F + B}{2}$.

And ſin. $\frac{AB + AF}{2}$: ſin. $\frac{AF \backsim AB}{2}$ :: *cotan.* $\frac{1}{2}A$: *tan.* $\frac{F \backsim B}{2}$.

Then $\frac{F + B}{2} + \frac{F \backsim B}{2} =$ *angle oppoſite* to the greater ſide.

And $\frac{F + B}{2} - \frac{F \backsim B}{2} =$ *angle op.* to the leſſer ſide.

Note, $\frac{F + B}{2}$ is of the ſame affection as $\frac{AF + AB}{2}$.

II. *Algebraically.*

Let s, a, d be the *ſines*, c, m, e the *coſines* of AB, A, AF; $z =$ *cotan.* AB, $t =$ *cotan.* AF, $\tau =$ *cotan.* A, *rad.* $= 1$. Then 47.

FIG.

$$\textit{Cotan.}\ B = \frac{se - dcm}{da} = \frac{s}{a}t - cr.$$

$$\textit{Cotan.}\ F = \frac{dc - sem}{as} = \frac{d}{a}z - er.$$

CASE VIII.

54. 55. 56. Given two *ſides* AF, AB, and the included *angle* A; to find the third *ſide* FB.

I. *Logarithmically.*

Let fall FI a perp. from F; then by Caſe 1 right ∠s,

Rad. : *coſ.* A : : *tan.* AF : *tan.* AI,

And AB ∽ AI = BI.

Then coſ. AI : *coſ.* AF : : *coſ.* BI : *coſ.* BF.

As BI and A are of the ſame or different affection, BF is < or > than a quadrant.

Or thus,

Find the ∠ B *by Caſe* 7; *then*

ſin. B : ſin. AF : : ſin. A : ſin. BF, *by Caſe* 4.

II. *Algebraically.*

47. *Let* s, d *be the ſines, and* c, e *the coſines of* AB, AF; m = *coſ.* A, v = *verſ.* A; D = *coſ.* AB ∽ AF; *rad.* = 1. *Then coſ.* FB = $sdm + ce$ = D − sdv.

CASE IX.

54. 55. 56. Given two *angles* A and F, and the included *ſide* AF; to find an oppoſite *ſide* FB.

I. *Logarithmically.*

Let fall a perp. from F; then by caſe 3ᵈ right ∠ triangles,

Rad. : *tan.* A : : *coſ.* AF : *cotan.* AFI;

And AFB ∽ AFI = BFI; then,

FIG.

$\textit{Coſ.}\ AFI : \textit{coſ.}\ BFI :: \textit{cotan.}\ AF : \textit{cotan.}\ BF.$

As BFI and A are of the ſame or different affection, BF is < or > than a quadrant. 54. 55. 56.

Or thus,

$$\textit{Coſ.}\ \frac{F + A}{2} : \textit{coſ.}\ \frac{F \backsim A}{2} :: \textit{tan.}\ \tfrac{1}{2}AF : \textit{tan.}\ \frac{AB + BF}{2}.$$

$$\text{And ſin.}\ \frac{F + A}{2} : \text{ſin.}\ \frac{F \backsim A}{2} :: \textit{tan.}\ \tfrac{1}{2}AF : \textit{tan.}\ \frac{AB \backsim BF}{2}.$$

Then $\frac{AB + BF}{2} + \frac{AB \backsim BF}{2} =$ *ſide* oppoſite to the greater angle.

And $\frac{AB + BF}{2} - \frac{AB \backsim BF}{2} =$ *ſide* opp. to the leſſer angle.

Note, $\frac{AF + BF}{2}$ is of the ſame affection as $\frac{F + A}{2}$.

II. *Algebraically.*

Let a, d, f be the ſines, and m, e, g the coſines of A, AF *and* F, $\tau =$ *cotan.* A, $t =$ *cotan.* AF, $y =$ *cotan.* F, *radius* $= 1$, then 47.

$$\textit{Cotan.}\ BF = \frac{f\tau + ge}{d} = \frac{f\tau}{d} + gt.$$

$$\textit{And cotan.}\ AB = \frac{ay + me}{d} = \frac{ay}{d} + mt.$$

CASE X.

Two *angles* A and F, and the included *ſide* AF, being given; to find the third *angle* B. 54. 55. 56.

I. *Logarithmically.*

FIG. 54. 55. 56. Let fall a perp. from F, then by Caſe 3 right angles,
Rad. : *tan.* A : : *coſ.* AF : *cotan.* AFI,
And AFB ∽ AFI = BFI ; then
ſin. AFI : ſin. BFI : : *coſ.* A : *coſ.* B.

As AFI is < or > than AFB, B and A are of the ſame or different affection.

Otherwiſe,

Find FB *by Caſe* 9 ; then B *by Caſe* 1.

II. *Algebraically.*

47. *Let a, f be the ſines, and m, g the coſines of* A *and* F, *e* = *coſ.* AF, *rad.* = 1 ; *then*
Coſ. B = *afe* — *mg*. *Alſo ſee Prop.* XLIII.

CASE XI.

54. 55. 56. Given the three *ſides*, to find an *angle* A.

I. *Logarithmically.*

Let fall a perp. from F ; then

Tan. ½AB : *tan.* $\frac{AF + FB}{2}$: : *tan.* $\frac{AF ∽ FB}{2}$: *tan.* an arch Q.

If ½AB > Q, the perpendicular falls within ; if leſs, without.

Then ½AB + Q = *greater ſegment* AI or BR, and ½AB ∽ Q = *leſſer ſegment* BI or AR ; then

If AF + FB < 180, the perpendicular falls neareſt the leſſer ſide.

If AF + FB > 180, the perpendicular falls neareſt the greater ſide.

Then by Caſe 4 *right* ∠s, *rad.* : *tan.* AI : : *cotan.* AF : *coſ.* A.

Or thus,

s. AB × s. AF : *rad. ſquare* :: s. $\frac{BF + AB - AF}{2}$ FIG. 54.
× s. $\frac{BF + AF - AB}{2}$: s. ſquare ½A. 55. 56.

Or s. AB × s. AF : *rad. ſquare* :: s. $\frac{BF + AB + AF}{2}$
× s. $\frac{BF + AB + AF}{2} - BF$: *coſ. ſquare* ½ A.

See alſo Prop. XLII. *and Cor.*

II. *Algebraically.*

Let d, s be the *ſines,* and *e, c* the *coſines of* AF, AB; *i* = *coſ.* FB, *z* = *coſ.* AB + AF, *x* = *coſ.* AB ∽ AF, *radius* = 1. 47.

Then coſ. A $= \frac{i - ce}{ds}$, *and verſ.* A $= \frac{x - i}{x - z} \times 2$.

Case XII.

The three *angles* being given; to find a *ſide* AF. 54. 55. 56.

I. *Logarithmically.*

As A, B are of the ſame or different affection; the perpendicular from F falls within or without.

Cot. $\frac{A + B}{2}$: *tan.* $\frac{A \backsim B}{2}$ :: *tan.* ½F : *tan. an arch* Q.

Then ½F + Q = *greater angle at the vertex* AFI or BFR.

And ½F ∽ Q = *leſſer angle at the vertex* BFI or AFR;

Then if A + B < 180, the perpendicular falls neareſt the leſſer ſide or greater angle.

If

FIG. If $A + B > 180$, the perpendicular falls neareſt
54. the greater ſide or leſſer angle.
55. *Then by Caſe* 16 *right angles,*
56. *Tan.* AFI : *cotan.* A : : *rad.* : *coſ.* AF.

Or thus,

Let $Z =$ ſum, $D =$ difference of A and F.
Then ſin. $A \times$ ſin. F : *rad. ſquare* : :
$\text{coſ.}\,\frac{B+Z}{2} \times \text{coſ.}\,\frac{B \backsim Z}{2}$: *ſine ſquare* ½AF.
Or ſin. $A \times$ ſin. F : *rad. ſquare* : :
$\text{coſ.}\,\frac{B+D}{2} \times \text{coſ.}\,\frac{B \backsim D}{2}$: *coſ. ſquare* ½AF.
Alſo ſee Prop. XLIII and Cor.

II. *Algebraically.*

47. *Let* a, f *be the ſines, and* m, g *the coſines of* A *and* F, $n = \text{coſ. } B$. $z = \text{coſ. } F + A$, $x = \text{coſ. } F \backsim A$, $rad = 1$. *Then*

$$\text{Coſ. } AF = \frac{mg + n}{af}, \text{ and verſ. } AF = \frac{n + z}{z - x} \times 2.$$

SCHOLIUM.

Any of the caſes of right-angled triangles is reſolved at one operation; but in obliques, all but the 1, 4, 11 and 12 require two operations, but then they find two things for theſe two operations, without a perpendicular. Let me only add, that the directions in general for performing the arithmetical work in ſpherics, is the ſame as in plain triangles.

The ſolution of all the caſes of ſpherical triangles in numbers. FIG.

Right-angled triangles.

CASE I.

Given hyp. AF = 78 20 } required the adjacent ſide AB. 79.
angle A = 27 43 }

Rad. — — — 10.
Coſ. A (27 43) — — 9.9470700
Tan. AF (78 20) — — 10.6851149
Tan. AB (76 52) — — 10.6321849

which is leſs than a quadrant, becauſe A and AF are of the ſame affection.

CASE II.

Given hyp. AF = 78 20 } required the oppoſite ſide BF.
angle A = 27 43 }

Rad. — — — 10.
Sin. AF (78 20) — — 9.9909338
Sin. A (27 43) — — 9.6675459
Sin. BF (27 06) — — 9.6584797

which is leſs than 90, becauſe A is acute.

CASE III.

FIG. 79. Given hyp. AF = 78 20 } required the other
angle A = 27 43 } ∠ F.

Rad.	—	—	10.
Coſ. AF (78 20)		—	9.3058189
Tan. A (27 43)	—	—	9.7204759
Cotan. F (83 56)	—	—	9.0262948

which is acute, becauſe AF and A are of the ſame affection.

CASE IV.

80. Given the hyp. AF = 78 20 } required the ∠
a leg AB = 76 52 } adjacent A.

Rad.	—	—	—	10.
Cot. AF (78 20)		—	—	9.3148851
Tan. AB (76 52)		—	—	10.6320468
Coſ. A (27 45)		—	—	9.9469319

which is acute, becauſe AB, AF are of the ſame affection.

CASE V.

80. Given hyp. AF = 78 20 } required the oppoſite
leg AB = 76 52 } ∠ F.

Sin. AF (78 20)	—	—	9.9909338
Rad.	—	—	10.
Sin. AB (76 52)	—	—	9.9884894
Sin. F (83 56)	—	—	9.9975556

which is acute, becauſe AB is leſs than 90.

CASE VI.

Given hyp. AF = 78 20 } required the other leg FIG.
leg. AB = 76 52 } BF. 80.

Coſ. AB (76 52) — —	9.3564426
Coſ. AF (78 20) — —	9.3058189
Rad. — — —	10.
Coſ. BF (27 08) — —	9.9493763

which is leſs than 90, becauſe AB, AF are of one affection.

CASE VII.

Given a ſide AB = 117 34 } required oppoſite 81.
adj. ∠ A = 31 51 } ſide BF.

Rad. — —	10.
Sin. AB (117 34) —	9.9476655
Tan. A (31 51) —	9.7932560
Tan. FB (28 51) —	9.7409215

which is leſs than 90, becauſe A is acute.

CASE VIII.

Given a ſide AB = 117 34 } required op. ∠ F. 81.
adj. ∠ A = 31 51 }

Rad. — —	10.
Sin. A (31 51) —	9.7223848
Coſ. AB (117 34) —	9.6653749
Coſ. F (104 08) —	9.3877597

which is obtuſe, becauſe AB is greater than 90.

CASE

CASE IX.

FIG. 81. Given a ſide AB = 117 34 } required hyp. AF.
adj. ∠ A = 31 51 }

Coſ. A	(31 51)	— —	9.9291289
Rad.	—	—	10.
Tan. AB	(117 34)	—	10.2822906
Tan. AF	(113 55)	—	10.3531617

which is greater than 90, becauſe A and AB are of different affection.

CASE X.

82. Given ſide BF = 28 51 } required ſide adj. AB.
op. ∠ A = 31 51 }

Rad.	—	—	10.
Cot. A	(31 51)	—	10.2067440
Tan. BF	(28 51)	—	9.7410662
Sin. AB	(62 28 or 117 32)	—	9.9478102

AB may be either leſſer or greater than 90.

CASE XI.

82. Given ſide BF = 28 51 } required adj. ∠ F.
op. ∠ A = 31 51 }

Coſ. BF	(28 51)	—	9.9424476
Coſ. A	(31 51)	—	9.9291289
Rad.	—	—	10.
Sin. F	(75 53 or 104 7)	—	9.9806813

which may be either acute or obtuſe.

CASE XII.

Given a ſide BF = 28 51 } required hyp. AF. FIG. 82.
op. ∠ A = 31 51 }

Sin. A (31 51)	—	9.7223848
Sin. BF (28 51)	—	9.6835137
Rad. —	—	10.
Sin. AF (66 07 or 113 53)	—	9.9611289

which may be either greater or leſs than 90.

CASE XIII.

Given, leg AB = 76 52 } required the ∠ A. 83.
leg FB = 27 06 }

Sin. AB (76 52)	— —	9.9884894
Rad.	— —	10.
Tan. BF (27 06)	— —	9.7090374
T. op. ∠ A (27 43)	—	9.7205480

which is acute, becauſe BF is leſs than 90.

CASE XIV.

Given the leg AB = 76 52 } required the hyp. 83.
leg FB = 27 06 } AF.

Rad.	— —	10.
Coſ. FB (27 06)	— —	9.9494938
Coſ. AB (76 52)	— —	9.3564426
Coſ. AF (78 20)	— —	9.3059364

which is leſs than a quadrant, becauſe AB, BF are of one affection,

CASE

CASE XV.

FIG. 84. Given the ∠ A = 31 51
∠ F = 104 08 } required a leg BF.

Sin. F (104 08) — — 9.9866509
Rad. — — 10.
Coſ. A (31 51) — — 9.9291289
Coſ. op. ſide BF (28 51) — 9.9424780
which is leſs than 90, becauſe A is acute.

CASE XVI.

84. Given ∠ A = 31 51
∠ F = 104 08 } required hyp. AF.

Tan. F (104 08) — — 10.5989422
Cot. A (31 51) — — 10.2067440
Rad. — — 10.
Coſ. AF (113 55) — — 9.6078018
which is greater than 90, becauſe A, F are of different affection.

* * *

Oblique ſpherical triangles.

CASE I.

Given AF = 79 13, BF = 62 42, ∠ A = 50 12 } required ∠ B. FIG. 85.

Sin. FB (62 42) — — 9.9487147
Sin. A (50 12) — — 9.8855215
Sin. AF (79 13) — — 9.9922626
19.8777841
Sin. B { 58 08 or 151 52 } — 9.9290694

which is ambiguous, becauſe A + B, and A + ſup. B are each of them leſs than 180.

CASE II.

Given AF = 79 13, BF = 62 42, ∠ A = 50 12 } required F. 85.

Draw FI perp. on AB, then
Rad — — 10.
Coſ. AF (79 13) — 9.2720635
Tan. A (50 12) — 10.0792671
Cot. AFI (77 21) — 9.3513306
Then
Cot. AF (79 13) — 9.2798009
Coſ. AFI (77 21) — 9.3404338
Cot. BF (62 42) — 9.7127662
19.0532000
Coſ. BFI (53 36) — 9.7733991

which is acute, becauſe BF and A are of the ſame affection. Then AFI ± BFI = F = 130 57 or 23 45, ambiguous.

CASE III.

FIG. 85.

Given AF = 79 13
BF = 62 42 } required AB.
∠ A = 50 12

Draw FI perp. on AB, then

Rad. — — 10.
Coſ. A (50 12) — 9.8062544
T. AF (79 13) — 10.7201991
T. AI (73 26) — 10.5264535

Then

Coſ. AF (79 13) — — 9.2710635
Coſ. AI (73 26) — — 9.4550441
Coſ. BF (62 42) — — 9.6614810
19.1165251
Coſ. BI (45 39) — — 9.8444616

which is leſs than 90, becauſe BF and A are of one affection. Then AI ∓ BI = AB = 27 47 or 119 5, ambiguous.

CASE IV.

86.

Given A = 50 12
B = 58 08 } required AF.
BF = 62 42

Sin. A (50 12) — — 9.8855215
Sin. BF (62 42) — — 9.9487147
Sin. B (58 08) — — 9.9290504
19.8777651
Sin. AF {79 12 / 100 48} — — 9.9922436

AF is ambiguous, becauſe BF + AF, and BF + ſup. AF, are both leſs than 180.

CASE V.

Given $\left.\begin{array}{r}A = 50\ 12\\ B = 58\ 08\\ BF = 62\ 42\end{array}\right\}$ required AB. FIG. 86.

Draw FI perp. to AB, then
Rad. — — 10.
Tan. BF (62 42) — — 10.2872338
Coſ. B (58 08) — — 9.7225881
Tan. BI (45 39) — — 10.0098219

Then

Cot. B (58 08) — — 9.7935378
Cot. A (50 12) — — 9.9207329
Sin. BI (45 39) — — 9.8543564
19.7750893
Sin. AI $\left\{\begin{array}{l}73\ 25\\ 106\ 35\end{array}\right\}$ — — 9.9815515

then AB (= AI + BI) = 119 04 or 152 14, ambiguous.

CASE VI.

Given $\left.\begin{array}{r}A = 50\ 12\\ B = 58\ 08\\ BF = 62\ 42\end{array}\right\}$ required F. 86.

Draw FI perp. on AB; then
Rad — — 10.
Tan. B (58 08) — — 10.2064622
Coſ. BF (62 42) — — 9.6614810
Cot. BFI (53 35) — — 9.8679432

Then

FIG. 86.	Coſ. B	(58 08)	— —	9.7225881
	Sin. BFI	(53 35)	— —	9.9056454
	Coſ. A	(50 12)	— —	9.8062544
				19.7118998
	Sin. AFI	(77 21 or 102 39)	—	9.9893117

Then F (= BFI + AFI) = 130 56 or 156 14, ambiguous.

CASE VII.

87. Given AF = 79 13, AB = 119 05, A = 50 12 } required B.

Draw FI perp. on AB; then

Rad.		— —	10.
Coſ. A	(50 12)	— —	9.8062544
Tan. AF	(79 13)	— —	10.7201991
Tan. AI	(73 26)	— —	10.5264535

Then AB — AI = IB = 45 39.

Sin. AI	(73 26)	— —	9.9815870
Sin. BI	(45 39)	— —	9.8543564
Cot. A	(50 12)	— —	9.9207329
			19.7750893
Cot. B	(58 08)	— —	9.7935023

which is acute, becauſe B is of the ſame affection as A.

CASE VIII.

Given AF = 79 13
AB = 119 05 required FB. FIG. 87.
A = 50 12

Draw FI perp. to AB; then
Rad. — — 10.
Coſ. A (50 12) — 9.8062544
Tan. AF (79 13) — 10.7201991
Tan. AI (73 26) — 10.5264535
Then AB — AI = BI = 45 39.
And coſ. AI (73 26) — 9.4550441
Coſ. AF (79 13) — 9.2720635
Coſ. BI (45 39) — 9.8445018
19.1165653
Coſ. BF (62 40) — 9.6615212

which is acute, becauſe BI and A are of the ſame affection.

CASE IX.

Given A = 50 12
AFB = 130 56 required FBI 88.
AF = 79 13

Draw FI perp. to AB; then
Rad. — — 10.
Tan. A (50 12) — — 10.0792671
Coſ. AF (79 13) — 9.2720635
Cot. AFI (77 21) — 9.3513306
Then AFB — AFI = BFI = 53 35.
Coſ. AFI (77 21) — 9.3404338
Coſ. BFI (53 35) — 9.7735327
Cot. AF (79 13) — 9.2798009
19.0533336
Cot. BF (62 42) — 9.7128998

which is leſs than 90, becauſe BFI and A are of the ſame affection.

CASE X.

FIG. 88.

$$\text{Given}\quad \left.\begin{array}{r} A = 50\ 12 \\ AFB = 130\ 56 \\ AF = 79\ 13 \end{array}\right\}\ \text{required}\ \angle B.$$

Draw FI perp. AB; then

Rad. — — 10.
Tan. A (50 12) — 10.0792671
Coſ. AF (79 13) — 9.2720635
Cot. AFI (77 21) — 9.3513306

Then AFB — AFI = BFI = 53 35.

Sin. AFI (77 21) — 9.9893279
Sin. BFI (53 35) — 9.9056454
Coſ. A (50 12) — 9.8062544
19.7118998
Coſ. B (58 08) — 9.7225719

which is acute, becauſe A is acute, and AFI leſs than AFB.

CASE XI.

89.

$$\text{Given}\quad \left.\begin{array}{r} AF = 79\ 13 \\ BF = 62\ 42 \\ AB = 119\ 05 \end{array}\right\}\ \text{required}\ A.$$

Draw FI perp. to AB; then

Tan. $\frac{1}{2}$AB (59 32$\frac{1}{2}$) — 10.2305741
Tan. $\frac{AF + BF}{2}$ (70 57$\frac{1}{2}$) — 10.4620031
Tan. $\frac{AF - BF}{2}$ (8 15$\frac{1}{2}$) — 9.1617917
19.6237948
Tan. Q = 13 54 — — 9.3932207

Then $\frac{1}{2}$AB + Q = AI = 73 26$\frac{1}{2}$.

And rad. — — 10.
Tan. AI (73 26$\frac{1}{2}$) — 10.5267740
Cot. AF (79 13) — 9.2798009
Coſ. A (50 10) — 9.8065749

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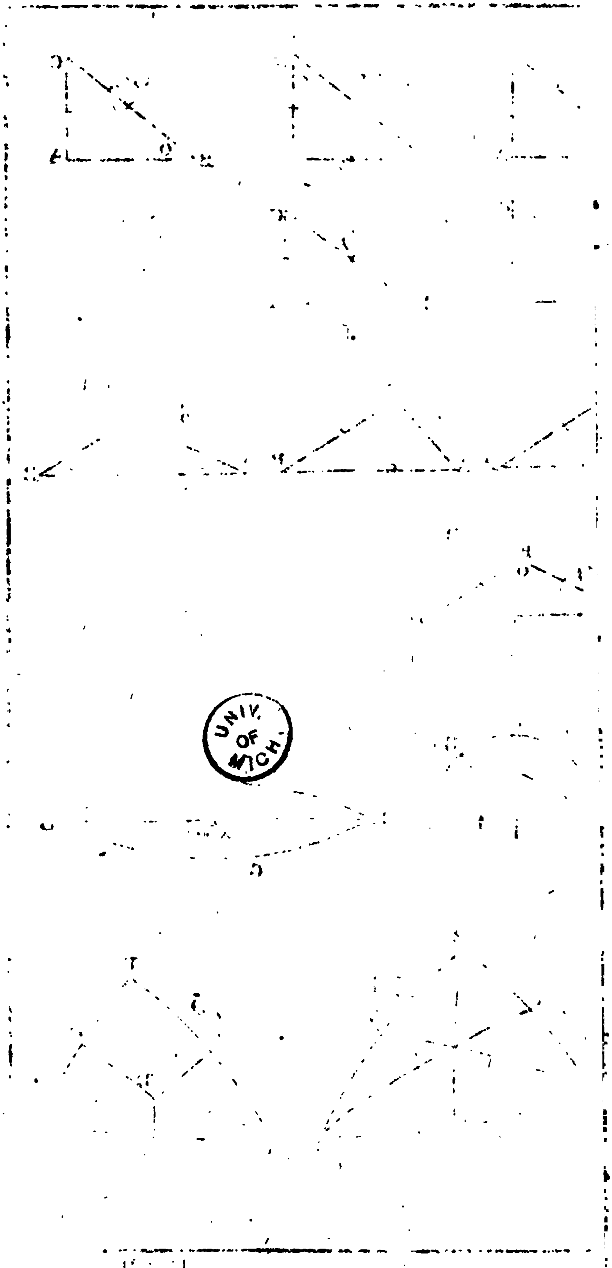

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CASE XII.

FIG. 90.

$$\left.\begin{aligned} \text{Given } A &= 50\ 12 \\ B &= 58\ 08 \\ F &= 130\ 56 \end{aligned}\right\}\text{ required } AF.$$

Draw FI perp. to AB; then

Cot. $\frac{1}{2}.\overline{A+B}$ (54 10) — 9.8586019
Tan. $\frac{1}{2}.\overline{B-A}$ (3 58) — 8.8409977
Tan. $\frac{1}{2}$ F (65 28) — 10.3406267
19.1816244
Tan. Q (11 53) — 9.3230225
Then $\frac{1}{2}F + Q = 77\ 21 = AFI$.
And tan. AFI (77 21) — 10.6488941
Cot. A (50 12) — 9.9207329
Rad. — — 10.
Cof. AF (79 13) — 9.2718388

F I N I S.